*Electronic Journal of Differential Equations*, Vol. 2008(2008), No. 108, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

# LARGE TIME BEHAVIOR OF SOLUTIONS TO SECOND-ORDER DIFFERENTIAL EQUATIONS WITH $p$-LAPLACIAN

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Abstract. We study asymptotic properties of solutions for certain second-order differential equation with $p$-Laplacian. The main purpose is to investigate when all global solutions behave at infinity like nontrivial linear functions. Making use of Bihari's inequality and its Dannan's version, we obtain results for differential equations with $p$-Laplacian analogous which extend those known in the literature concerning ordinary second order differential equations.

## 1. Introduction

In this paper, we study asymptotic properties of the second-order differential equation with $p$-Laplacian

$$(|u'|^{p-1}u')' + f(t,u,u') = 0, \quad p \geq 1. \tag{1.1}$$

In the sequel, it is assumed that all solutions of (1.1) are continuously extendable throughout the entire real axis. We refer to such solutions as to global solutions. We shall prove sufficient conditions under which all global solutions are asymptotic to $at+b$, as $t \to +\infty$, where $a, b$ are real numbers. The problem for ordinary second order differential equations without $p$-Laplacian has been studied by many authors, e. g. by Cohen [6], Constantin [7], Dannan [8], Kusano and Trench [9, 10], Rogovchenko [13], Rogovchenko [14], Tong [15] and Trench [16]. Our results are more close to those obtained in the papers [13, 14]. The main tool of the proofs are the Bihari's and Dannan's integral inequalities. We remark that sufficient conditions on the existence of global solutions for second order differential equations and second order functional-differential equations with $p$-Laplacian are proved in the papers [1, 2, 3, 4, 11]. Many references concerning differential equations with $p$-Laplacian can be found in the paper by Rachunková, Staněk and Tvrdý [12], where boundary value problems for such equations are treated.

Let

$$u(t_0) = u_0, \quad u'(t_0) = u_1, \tag{1.2}$$

where $u_0, u_1 \in \mathbb{R}$ be initial condition for solutions of (1.1).

We say that a solution $u(t)$ of (1.1) possesses the property (L) if $u(t) = at+b+o(t)$ as $t \to \infty$, where $a$, $b$ are real constants.

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2000 *Mathematics Subject Classification.* 34C11.
*Key words and phrases.* Second order differential equation; $p$-Laplacian; Bihari's inequality; asymptotic properties; Dannan's inequality.

Submitted June 9, 2008. Published August 11, 2008.

## 2. Main results

**Theorem 2.1.** *Let $p \geq 1$, $r > 0$ and $t_0 > 0$. Suppose that the following conditions are satisfied:*

(1) *$f(t,u,v)$ is a continuous function in $D = \{(t,u,v) : t \in < t_0, \infty), u, v \in \mathbb{R}\}$, where $t_0 > 0$*
(2) *There exist continuous functions $h, g : \mathbb{R}_+ = < 0, \infty) \to \mathbb{R}_+$ such that*
$$|f(t,u,v)| \leq h(t) g\Big(\big[\frac{|u|}{t}\big]^r\Big)|v|^r,\ (t,u,v) \in D,$$
*where for $s > 0$ the function $g(s)$ is positive and nondecreasing,*
$$\int_{t_0}^{\infty} h(s)\mathrm{d}s < \infty,$$
*and if we denote*
$$G(x) = \int_{t_0}^{x} \frac{\mathrm{d}s}{s^{r/p} g(s^{r/p})},$$
*then*
$$G(\infty) = \int_{t_0}^{\infty} \frac{\mathrm{d}s}{s^{r/p} g(s^{r/p})} = \frac{p}{r} \int_{a}^{\infty} \frac{\tau^{\frac{p}{r}-1}\mathrm{d}\tau}{\tau g(\tau)} = \infty,$$
*where $a = (t_0)^{r/p}$.*

*Then any global solution $u(t)$ of the equation (1) possesses the property (L).*

*Proof.* Without loss of generality we may assume $t_0 = 1$. Let $u(t)$ be a solution of (1.1), (1.2). Then
$$(u'(t))^p \leq |u'(t)|^{p-1}u'(t) \leq c_2 + \int_1^t |f(s,u(s),u'(s))|\mathrm{d}s, \tag{2.1}$$
where $c_2 = |u_1|^p$. Let $w(t)$ be the right-hand side of inequality (2.1). Then
$$u'(t) \leq w(t)^{1/p}$$
and
$$u(t) \leq c_1 + \int_1^t w(s)^{1/p}\mathrm{d}s \leq c_1 + (t-1)w(t)^{1/p} \leq t[c_1 + w(t)^{1/p}], \tag{2.2}$$
where $c_1 = |u_0|$, i.e.
$$u(t) \leq t[c_1 + w(t)^{1/p}], t \geq 1.$$
Applying the inequality $(A+B)^p \leq 2^{p-1}(A^p + B^p), A, B \geq 0$ and the assumption (2) of Theorem 2.1 we obtain from (2.2):
$$\begin{aligned}\Big(\frac{|u(t)|}{t}\Big)^p &\leq 2^{p-1}c_1^p + 2^{p-1}w(t)\\ &\leq 2^{p-1}c_1^p + 2^{p-1}\Big(c_2 + \int_0^t h(s)g\Big(\big[\frac{|u(s)|}{s}\big]^r\Big)|u'(s)|^r\Big)\mathrm{d}s.\end{aligned} \tag{2.3}$$
Let
$$d = 2^{p-1}(c_1^p + c_2), \quad H(t) = 2^{p-1}h(t). \tag{2.4}$$
Then
$$\Big(\frac{|u(t)|}{t}\Big)^p \leq d + \int_1^t H(s)g\Big(\big[\frac{|u(s)|}{s}\big]^r\Big)|u'(s)|^r\mathrm{d}s := z(t); \tag{2.5}$$

i.e.,

$$\Big(\frac{|u(t)|}{t}\Big)^r \le z(t)^{r/p}.$$

From the assumption (2) of Theorem 2.1 and the inequality (2.1) it follows

$$|u'(t)|^p \le u_1^p + \int_1^t h(s)g\Big(\big[\frac{|u(s)|}{s}\big]^r\Big)|u'(s)|^r \mathrm{d}s \le z(t);$$

i.e. we have

$$|u'(t)|^p \le z(t).$$

Since $g(s)$ is nondecreasing, the inequality (2.3) yields

$$g\Big(\big[\frac{|u(t)|}{t}\big]^r\Big) \le g(z(t)^{r/p})$$

and so we conclude for $t \ge 1$,

$$z(t) \le d + \int_1^t H(s)g(z(t)^{r/p})z(t)^{r/p}\mathrm{d}s.$$

From the assumption (2) of Theorem 2.1 it follows that the inverse $G^{-1}$ of $G$ is defined on the interval $(G(+0), \infty)$. Applying the Bihari theorem (see [5]) we obtain

$$z(t) \le G^{-1}\Big(G(d) + 2^{p-1}\int_1^\infty h(s)\mathrm{d}s\Big) := K < \infty.$$

Therefore the inequality (2.4) yields

$$|u'(t)| \le L := K^{1/p}$$

and from (2.3) we have

$$\frac{|u(t)|}{t} \le L.$$

Since

$$\int_1^t |f(s, u(s), u'(s))|\mathrm{d}s \le \int_1^t h(s)g(\Big(\frac{|u(s)|}{s}\Big)^r)|u'(s)|^r\mathrm{d}s \le z(t) \le K$$

for $t \ge 1$, the integral $\int_1^\infty |f(s, u(s), u'(s))|\mathrm{d}s$ exists. From (2.5) it follows that there exists $a \in \mathbb{R}$ such that

$$\lim_{t\to\infty} u'(t) = a.$$

By the l'Hospital rule, we can conclude that

$$\lim_{t\to\infty} \frac{u(t)}{t} = \frac{u_1 + \int_1^t u'(\tau)\mathrm{d}\tau}{t} = \lim_{t\to\infty} u'(t) = a.$$

Therefore there exist $b \in \mathbb{R}$ such that $u(t) = at + b + o(t)$. $\square$

**Example 1.** Let $t_0 = 1$, $p \ge r > 0$,

$$f(t,u,u') = \eta(t)t^{1-\alpha}e^{-t}\Big(\frac{u}{t}\Big)^{p-r}\ln\Big[2+\Big(\frac{|u|}{t}\Big)^r\Big](u')^r,\ t\ge 1, \tag{2.6}$$

where $0 < \alpha < 1$, $\eta(t)$ is a continuous function on interval $\langle 1,\infty)$ with $K = \sup_{t\ge 1}|\eta(t)| < \infty$.

The function $f(t,u,u')$ can be written in the form

$$f(t,u,u') = h(t)g\Big([\frac{u}{t}]^r\Big)(u')^r, \tag{2.7}$$

where $h(t) = \eta(t)t^{1-\alpha}e^{-t}$, $g(u) = u^{\frac{p}{r}-1}\ln(2+|u|)$. Obviously $g(u)$ is positive, continuous and nondecreasing function, $\int_1^\infty |h(s)|ds < K\Gamma(\alpha) = K\int_0^\infty s^{1-\alpha}e^{-s}\mathrm{d}s$ and

$$\int_1^\infty \frac{\tau^{\frac{p}{r}-1}\mathrm{d}\tau}{\tau g(\tau)} = \int_1^\infty \frac{\mathrm{d}\tau}{\tau\ln(2+\tau)} > \int_1^\infty \frac{\mathrm{d}\tau}{(2+\tau)\ln(2+\tau)} = \infty. \tag{2.8}$$

Thus we have proved that all conditions of Theorem 1 are satisfied. This means that for every solution $u(t)$ of the initial value problem (1.1), (1.2) there exist numbers $a, b$ such that $u(t) = at + b + o(t)$ as $t \to \infty$.

**Theorem 2.2.** *Let $p \ge 1, r > 0$ and $t_0 > 0$. Suppose the following conditions are satisfied:*

(1) *The function $f(t,u,v)$ is continuous in $\mathrm{D} = \{(t,u,v) : t \in < t_0,\infty), u, v \in \mathbb{R}\}$,*
(2) *There exist continuous functions $h_1, h_2, h_3, g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$|f(t,u,v)| \le h_1(t)g_1\Big([\frac{|u|}{t}]^r\Big) + h_2(t)g_2(|v|^r) + h_3(t),\ (t,u,v) \in D,$$

*where $H_i := \int_{t_0}^\infty h_i(s)\mathrm{d}s < \infty$, $i = 1,2,3$, for $s > 0$ the functions $g_1(s)$, $g_2(s)$ are nondecreasing and if*

$$G(x) = \int_{t_0}^x \frac{\mathrm{d}s}{g_1(s^{r/p}) + g_2(s^{r/p})}$$

*then*

$$G(\infty) = \int_{t_0}^\infty \frac{\mathrm{d}s}{g_1(s^{r/p}) + g_2(s^{r/p})} = \frac{p}{r}\int_a^\infty \frac{\tau^{\frac{p}{r}-1}\mathrm{d}\tau}{g_1(\tau) + g_2(\tau)} = \infty,$$

*where $a = (t_0)^{r/p}$.*

*Then any global solution $u(t)$ of the equation (1) possesses the property (L).*

*Proof.* Without loss of generality we may assume $t_0 = 1$. By the standard existence results, it follows from the continuity of the function $f$ that equation (1.1) has solution $u(t)$ corresponding to the initial data $u(1) = u_0$, $u'(1) = u_1$. Two times of integration (1.1) from 1 to $t$, yields for $t \ge 1$

$$(u'(t))^p \le |u'(t)|^{p-1}u'(t) = u_1^p - \int_1^t f(s,u(s),u'(s))\mathrm{d}s, \tag{2.9}$$

$$u(t) \le u_0 + (t-1)\Big[u_1^p - \int_1^t f(s,u(s),u'(s))\mathrm{d}s\Big]^{1/p}. \tag{2.10}$$

It follows from (2.9) and (2.10) that for $t \geq 1$,

$$
\begin{gathered}
|u'(t)| \leq w(t)^{1/p}, \\
|u(t)| \leq t\big(c_1 + w(t)^{1/p}\big),
\end{gathered}
$$

where $c_1 = |u_0|$, $c_2 = |u_1|^p$, $w(t) = c_2 + \int_1^t |f(s, u(s), u'(s))| \mathrm{d}s$. Using the assumption (2) we obtain for $t \geq 1$

$$
\begin{aligned}
|u'(t)| \leq \Big[c_2 + \int_1^t h_1(s) g_1\Big(\big[\frac{|u(s)|}{s}\big]^r\Big)\mathrm{d}s \\
+ \int_1^t h_2(s) g_2(|u'(s)|^r)\mathrm{d}s + \int_1^t h_3(s)\mathrm{d}s\Big]^{1/p}, \\
\frac{|u(t)|}{t} \leq c_1 + \Big[c_2 + \int_1^t h_1(s) g_1\Big(\big[\frac{|u(s)|}{s}\big]^r\Big)\mathrm{d}s \\
+ \int_1^t h_2(s) g_2(|u'(s)|^r)\mathrm{d}s + \int_1^t h_3(s)\mathrm{d}s\Big]^{1/p}.
\end{aligned}
$$

Applying the inequality $(A+B)^p \leq 2^{p-1}(A^p + B^p)$, where $A, B \geq 0$, we obtain

$$
\begin{aligned}
(\frac{|u(t)|}{t})^p \leq d + \int_1^t H_1(s) g_1\Big(\big[\frac{|u(s)|}{s}\big]^r\Big)\mathrm{d}s \\
+ \int_1^t H_2(s) g_2(|u'(s)|^r)\mathrm{d}s + \int_1^t H_3(s)\mathrm{d}s.
\end{aligned} \tag{2.11}
$$

where $d = 2^{p-1}(c_1^p + c_2)$, $H_i(t) = 2^{p-1}h_i(t)$, $i = 1, 2, 3$. Denote by $z(t)$ the right-hand side inequality (2.11)

$$
|u'(t)|^r \leq z(t)^{r/p}, \quad \Big(\frac{|u(t)|}{t}\Big)^r \leq z(t)^{r/p}. \tag{2.12}
$$

Since the function $g_1(s)$ and $g_2(s)$ are nondecreasing for $s > 0$, we obtain

$$
g_1\Big(|u'(t)|^r\Big) \leq g_1\Big(z(t)^{r/p}\Big), \quad g_1\Big(\Big[\frac{|u(t)|}{t}\Big]^r\Big) \leq g_2\Big(z(t)^{r/p}\Big).
$$

Thus, for $t \geq 1$,

$$
z(t) \leq d + \int_1^t H_1(s) g_1(z(s)^{r/p})\mathrm{d}s + \int_1^t H_2(s) g_2(z(s)^{r/p})\mathrm{d}s + \int_1^t H_3(s)\mathrm{d}s. \tag{2.13}
$$

Furthermore, due to evident inequality

$$
H_1(s) g_1(z(s)^{r/p}) + H_2(s) g_2(z(s)^{r/p}) \leq (H_1(s) + H_2(s))(g_1(z(s)^{r/p}) + g_2(z(s)^{r/p})) \tag{2.14}
$$

By (2.14), we have

$$
z(t) \leq d + \bar{H}_3 + \int_1^t (H_1(s) + H_2(s))(g_1(z(s)^{r/p}) + g_2(z(s)^{r/p}))\mathrm{d}s;
$$

i.e.,

$$
z(t) \leq d + 2^{p-1}\bar{h}_3 + 2^{p-1}\int_1^t (h_1(s) + h_2(s))(g_1(z(s)^{r/p}) + g_2(z(s)^{r/p}))\mathrm{d}s. \tag{2.15}
$$

Applying Bihari's inequality (see [5]) to (2.15), we obtain, for $t \geq 1$,

$$
z(t) \leq G^{-1}\Big(G(d + 2^{p-1}\bar{h}_3) + 2^{p-1}\int_1^t (h_1(s) + h_2(s))\mathrm{d}s\Big),
$$

where
$$G(x)=\int_1^x \frac{\mathrm{d}s}{g_1(s^{r/p})+g_2(s^{r/p})},$$
and $G^{-1}(x)$ is the inverse function for $G(x)$ defined for $x\in(G(+0),\infty)$. Note that $G(+0)<0$, and $G^{-1}(x)$ is increasing.

Now, let
$$K=G(d+2^{p-1}\bar{h}_3)+2^{p-1}(\bar{h}_1+\bar{h}_2)<\infty.$$
Since $G^{-1}(x)$ is increasing, we have
$$z(t)\le G^{-1}(K)<\infty;$$
so it yields
$$\frac{|u(t)|}{t}\le G^{-1}(K),\quad |u'(t)|\le G^{-1}(K).$$
Using assumption (2) of the Theorem 2.2, we have
$$\begin{aligned}\int_1^t |f(s,u(s),u'(s))|\mathrm{d}s &\le h_1(t)g_1\Big([\frac{|u|}{t}]^r\Big)+h_2(t)g_2(|u'(s)|^r)+h_3(t)\\ &\le z(t)\le G^{-1}(K),\end{aligned}$$
where $t\ge 1$, the integral $\int_1^t |f(s,u(s),u'(s))|\mathrm{d}s$ converges, and there exists an $a\in\mathbb{R}$ such that
$$\lim_{t\to\infty} u'(t)=a.$$
□

**Example 2.** Let $t_0=1, p\ge r>0$,
$$\begin{aligned}f(t,u,v)&=\eta_1(t)t^{1-\alpha_1}e^{-t}\big(\frac{u}{t}\big)^{p-r}\ln\big[2+\big(\frac{u}{t}\big)^r\big]\\ &\quad+\eta_2(t)t^{1-\alpha_2}e^{-t}v^{p-r}\ln(3+v^r)+\eta_3(t)t^{1-\alpha_3}e^{-t}\end{aligned}$$
where $0<\alpha_i<1$, $\eta_i(t)$ are continuous functions on $[1,\infty)$, $K_i=\sup_{t\ge1}|\eta_i(t)|<\infty$, $i=1,2,3$. Then $f(t,u,u')$ can be written as
$$f(t,u,v)=h_1(t)g_1\big([\frac{u}{t}]^r\big)+h_2(t)g_2(v^r)+h_3(t),$$
where $h_i(t)=\eta_i(t)t^{1-\alpha_i}e^{-t}$, $i=1,2,3$, $g_1(u)=u^{\frac{p}{r}}\ln(2+u)$, $g_2(u)=u^{\frac{p}{r}}\ln(2+u)$. Then
$$|f(t,u,v)|\le|h_1(t)|g_1\Big([\frac{u}{t}]^r\Big)+|h_2(t)|g_2(|v|^r)+|h_3(t)|,$$
where $(t,u,v)\in D=\{(t,u,v):t\in\langle1,\infty),u,v\in\mathbb{R}\}$, $|h_i(t)|\le K_i\Gamma(\alpha_i)$, $i=1,2,3$ and obviously we have
$$\begin{aligned}G(\infty)&=\int_1^\infty\frac{\tau^{\frac{p}{r}-1}\mathrm{d}\tau}{g_1(\tau)+g_2(\tau)}\\ &=\int_1^\infty\frac{\tau^{\frac{p}{r}-1}\mathrm{d}\tau}{\tau^{\frac{p}{r}}[\ln(2+\tau)+\ln(3+\tau)]}\\ &\ge\frac12\int_1^\infty\frac{\mathrm{d}\tau}{(3+\tau)\ln(3+\tau)}=\infty.\end{aligned}$$
This means that all assumptions of Theorem 2.2 are satisfied and thus any global solution $u(t)$ of the equation (1) possesses the property (L).

**Theorem 2.3.** *Let $t_0>0$. Suppose that the following assumptions hold:*

(i) *there exist nonnegative continuous function* $h_1, h_2, g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*
$$|f(t,u,v)| \le h_1(t) g_1\Big( [\frac{|u|}{t}]^r \Big) + h_2(t) g_2(|v|^r);$$
(ii) *for* $s > 0$ *the function* $g_1(s)$, $g_2(s)$ *are nondecreasing, and*
$$g_1(\alpha u) \le \psi_1(\alpha) g_1(u), \qquad g_2(\alpha u) \le \psi_2(\alpha) g_2(u)$$
*for* $\alpha \ge 1$, $u \ge 0$, *where the functions* $\psi_1(\alpha)$, $\psi_2(\alpha)$ *are continuous for* $\alpha \ge 1$;

(iii) $\int_{t_0}^{\infty} h_i(s) \mathrm{d}s = H_i < \infty$, $i = 1, 2$.

*Assume that there exists a constant* $K \ge 1$ *such that*
$$\begin{aligned} K^{-1}(\psi_1(K) + \psi_2(K)) 2^{p-1}(H_1 + H_2) &\le \int_{t_0}^{+\infty} \frac{\mathrm{d}s}{g_1(s^{r/p}) + g_2(s^{r/p})} \\ &= \frac{p}{r} \int_a^{+\infty} \frac{\tau^{\frac{p}{r}-1} \mathrm{d}\tau}{g_1(\tau) + g_2(\tau)}, \end{aligned}$$
*where* $a = (t_0)^{r/p}$. *Then any global solution* $u(t)$ *of the equation* (1.1) *with initial data* $u(t_0) = u_0$, $u'(t_0) = u_1$ *such that* $(|u_0| + |u_1|)^p \le K$ *possesses the property (L).*

*Proof.* Without loss of generality we may assume $t_0 = 1$. Arguing in the same way as in Theorem 2.1, we obtain by assumption (i) of Theorem 2.3
$$|u'(t)| \le \Big[ |u_1|^p + \int_1^t h_1(s) g_1\Big( [\frac{u(s)}{s}]^r \Big) \mathrm{d}s + \int_1^t h_2(s) g_2(|u'(s)|^r) \mathrm{d}s \Big]^{1/p} \tag{2.16}$$
$$\frac{|u(t)|}{t} \le |u_0| + \Big[ |u_1|^p + \int_1^t h_1(s) g_1\Big( [\frac{u(s)}{s}]^r \Big) \mathrm{d}s + \int_1^t h_2(s) g_2(|u'(s)|^r) \mathrm{d}s \Big]^{1/p} \tag{2.17}$$
where $t \ge 1$.
$$\Big( \frac{|u(t)|}{t} \Big)^p \le K + 2^{p-1} \Big( \int_1^t h_1(s) g_1\Big( [\frac{u(s)}{s}]^r \Big) \mathrm{d}s + \int_1^t h_2(s) g_2(|u'(s)|^r) \mathrm{d}s \Big), \tag{2.18}$$
where $K = 2^{p-1}(|u_0|^p + |u_1|^p) \ge (|u_0| + |u_1|)^p$. Denoting by $z(t)$ the right-hand side of inequality (2.18) we have by (2.16) and (2.18)
$$|u'(t)|^r \le z(t)^{r/p}, \quad \Big( \frac{|u(t)|}{t} \Big)^r \le z(t)^{r/p}. \tag{2.19}$$
Since the function $g_1(s)$, $g_2(s)$ are nondecreasing for $s > 0$, for $t \ge 1$, (2.19) yields
$$z(t) \le K + 2^{p-1} \Big( \int_1^t h_1(s) g_1\big(z(s)^{r/p}\big) \mathrm{d}s + \int_1^t h_2(s) g_2\big(z(s)^{r/p}\big) \Big) \mathrm{d}s. \tag{2.20}$$
By assumption (ii) of Theorem 2.3, the functions $g_1(u)$, $g_2(u)$ belong to the class $\mathbb{H}$. Furthermore, if $g_1(u)$ and $g_2(u)$ belong to the class $\mathbb{H}$ with corresponding multiplier function $\psi_1(\alpha)$, $\psi_2(\alpha)$ respectively, then the sum $g_1(u) + g_2(u)$. Applying Bihari's Theorem (see [5]) to (2.20), we have for $t \ge 1$
$$z(t) \le K W^{-1}(K^{-1}(\psi_1(K) + \psi_2(K))) 2^{p-1} \int_1^t (h_1(s) + h_2(s)) \mathrm{d}s, \tag{2.21}$$
where
$$W(u) = \int_1^u \frac{\mathrm{d}s}{g_1\big(s^{r/p}\big) + g_2\big(s^{r/p}\big)},$$

and $W^{-1}(u)$ is inverse function for $W(u)$. Inequality (2.21) holds for all $t \geq 1$ because

$$(K^{-1}(\psi_1(K)+\psi_2(K))2^{p-1}(H_1+H_2) = L < \infty.$$

Since $W^{-1}(u)$ is increasing, we get

$$z(t) \leq KW^{-1}(L) < \infty,$$

so it follows from (2.19), (2.20) that

$$\frac{|u(t)|}{t} \leq KW^{-1}(L), \quad |u'(t)| \leq KW^{-1}(L).$$

The rest of the proof is similar to that of Theorem 2.2 and thus it is omitted. □

**Example 3.** Let $t_0 > 0$. Consider (1.1) with $p \geq 1$, $\frac{p}{q} = 2$,

$$f(t,u,v) = h_1(t)u^2 = h_2(t)v^2, \tag{2.22}$$

where $h_1(t) = \frac{\eta_1(t)}{t^2} t^{1-\alpha_1} e^{-t}$, $h_2(t) = \eta_2(t) t^{1-\alpha_2} e^{-t}$, $0 < \alpha_i \leq 1$, $\eta_i(t)$, $i = 1,2$ are continuous functions on the interval $\langle 0,\infty)$ with $K_i = \sup_{t\geq t_0} |\eta_i(t)| < \infty$. Then we can write

$$f(t,u,v) = \eta_1(t)t^{1-\alpha_1}e^{-t}\big(\frac{u}{t}\big)^2 + \eta_2(t)t^{1-\alpha_2}e^{-t}v^2 \tag{2.23}$$

and

$$|f(t,u,u')| \leq K_1\Gamma(\alpha_1)g_1(u) + K_2\Gamma(\alpha_2)g_2(u'), \tag{2.24}$$

where $g_1(u) = u^2$, $g_2(u') = (u')^2$ . The functions $g_1, g_2$ satisfy the condition (ii) of Theorem 2.3 with $\psi_1(\alpha) = \psi_2(\alpha) = \alpha^2$ and

$$\int_{t_0}^{\infty} \frac{\tau^{\frac{p}{r}-1}\mathrm{d}\tau}{g_1(\tau)+g_2(\tau)} = \int_{t_0}^{\infty} \frac{\mathrm{d}\tau}{\tau} = \infty. \tag{2.25}$$

Thus all assumptions of Theorem 2.3 are satisfied and therefore any global solution $u(t)$ of the equation (1.1) (independently on the initial values $u_0, u_1$) possesses the property (L).

**Theorem 2.4.** *Let $t_0 > 0$. Suppose that the assumptions (i) and (iii) of Theorem 2.3 hold, while (ii) is replaced by*

- (ii') *for $s > 0$ the functions $g_1(s)$, $g_2(s)$ are nonnegative, continuous and non-decreasing, $g_1(0) = g_2(0) = 0$ and satisfy a Lipschitz condition*
$$|g_1(u+v) - g_1(u)| \leq \lambda_1 v, \quad |g_2(u+v) - g_2(u)| \leq \lambda_2 v,$$
*where $\lambda_1, \lambda_2$ are positive constants.*

*Then any global solution $u(t)$ of (1.1) with initial data $u(t_0) = u_0$, $u'(t_0) = u_1$ such that $|u_0|^p + |u_1|^p \leq K$ possesses property (L).*

*Proof.* Applying [8, Corollary 2] to (2.20), we have for $t \geq 1$

$$\begin{aligned}
z(t) &\leq K + 2^{p-1}\int_{t_0}^{t} (h_1(s)+h_2(s))(g_1(K)+g_2(K)) \\
&\quad \times \exp\Big(2^{p-1}\int_{t_0}^{t} (\lambda_1+\lambda_2)(h_1(\tau)+h_2(\tau))\mathrm{d}\tau\Big)\mathrm{d}s \\
&\leq K + 2^{p-1}(H_1+H_2)(g_1(K)+g_2(K))\exp\Big(2^{p-1}(\lambda_1+\lambda_2)(H_1+H_2)\Big) \\
&< +\infty.
\end{aligned}$$

The proof can be completed with the same argument as in Theorem 2.2. □

**Theorem 2.5.** *Let $t_0 > 0$. Suppose that there exist continuous functions $h, g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$|f(t,u,v)| \le h(t) g_1\Big([\frac{|u|}{t}]^r\Big) g_2(|v|^r),$$

*where for $s > 0$ the functions $g_1(s)$, $g_2(s)$ are nondecreasing;*

$$\int_{t_0}^{\infty} h(s)\mathrm{d}s < \infty,$$

*and if we denote*

$$G(x) = \int_{t_0}^{x} \frac{\mathrm{d}s}{g_1(s^{r/p}) g_2(s^{r/p})}$$

*then $G(+\infty) = \frac{p}{r}\int_a^\infty \frac{\tau^{\frac{p}{r}-1}}{g_1(\tau)g_2(\tau)}\mathrm{d}\tau = +\infty$, where $a = (t_0)^{\frac{r}{p}}$. Then any global solution $u(t)$ of the equation (1.1) possesses the property (L).*

*Proof.* Without loss of generality we may assume $t_0 = 1$. Arguing as in the proof of Theorem 2.2, we obtain for $t \ge 1$

$$\begin{gathered} |u'(t)| \le \Big[|u_1|^p + \int_1^t h(s) g_1\Big([\frac{u(s)}{s}]^r\Big) g_2(|u'(s)|^r)\mathrm{d}s\Big]^{1/p}, \\ \frac{|u(t)|}{t} \le |u_0| + \Big[|u_1|^p + \int_1^t h(s) g_1\Big([\frac{u(s)}{s}]^r\Big) g_2(|u'(s)|^r)\mathrm{d}s\Big]^{1/p}, \\ \Big(\frac{|u(t)|}{t}\Big)^p \le C + 2^{p-1}\int_1^t h(s) g_1\Big([\frac{u(s)}{s}]^r\Big) g_2(|u'(s)|^r)\mathrm{d}s, \end{gathered} \tag{2.26}$$

where $C = 2^{p-1}(|u_0|^p + |u_1|^p) \ge (|u_0| + |u_1|)^p$. Denoting by $z(t)$ the right-hand side of inequality (2.26) and using the assumptions of the Theorem 2.5, we have for $t \ge 1$

$$z(t) \le 1 + C + 2^{p-1}\int_1^t h(s) g_1(z^{r/p}) g_2(z^{r/p})\mathrm{d}s. \tag{2.27}$$

Applying Bihari's inequality (see [5]) to (2.27), for $t \ge 1$, we obtain

$$z(t) \le G^{-1}\Big(G(1+C) + 2^{p-1}\int_1^t h(s)\mathrm{d}s\Big) \le G^{-1}(K),$$

where

$$G(w) = \int_1^w \frac{\mathrm{d}s}{g_1(s^{r/p}) g_2(s^{r/p})},$$

and $G^{-1}(w)$ is the inverse function for $G(w)$. The function $G^{-1}(w)$ is defined for $w \in (G(+0), \infty)$, where $G(+0) < 0$, it is increasing, and

$$K = G(1+C) + 2^{p-1}\int_1^\infty h(s)\mathrm{d}s < \infty.$$

The rest of proof is similar that of Theorem 2.2 and thus is omitted. $\square$

**Example 4.** Let $t_0 = 1$, $p \geq r > 0$,

$$f(t,u,v) = \eta(t)t^{1-\alpha}e^{-t}\Big[\Big(\frac{u}{t}\Big)^{p-r}\ln\big[2+\Big(\frac{u}{t}\Big)^{r}\big]\Big]^{\frac{3}{4}} \cdot \Big[v^{p-r}\ln(2+v^r)\Big]^{\frac{1}{4}},$$

where $\eta(t)$ is a continuous function on $\langle 1,\infty)$ with $K = \sup_{t\in\langle 1,\infty)}\eta(t) < \infty$. Let

$$g_1(u) = \Big[u^{\frac{p}{r}-1}\ln(2+u)\Big]^{3/4}, \quad g_2(v) = \Big[v^{\frac{p}{r}-1}\ln(2+v)\Big]^{1/4}, \quad h(t) = \eta(t)t^{1-\alpha}e^{-t}.$$

Then

$$f(t,u,v) = h(t)g_1\Big([\frac{u}{t}]^r\Big)g_2(v^r)$$

and

$$\begin{aligned} G(+\infty) &= \frac{p}{r}\int_1^\infty \frac{\tau^{\frac{p}{r}-1}}{g_1(\tau)g_2(\tau)}\mathrm{d}\tau = \frac{p}{r}\int_1^\infty \frac{\mathrm{d}\tau}{\tau\ln(2+\tau)} \\ &> \frac{p}{r}\int_1^\infty \frac{\mathrm{d}\tau}{(2+\tau)\ln(2+\tau)} = +\infty. \end{aligned}$$

Obviously $|f(t,u,v)|$ can be estimated as in Theorem 2.5. Thus all assumptions of Theorem 2.5 are satisfied and this means that any global solution of the equation (1.1) possesses the property (L).

**Theorem 2.6.** *Let $t_0 > 0$. Suppose that the following conditions hold:*

(i) *there exist nonnegative continuous functions $h, g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$|f(t,u,v)| \leq h(t)g_1\Big(\Big[\frac{|u(t)|}{t}\Big]^r\Big)g_2(|v|^r)$$

(ii) *for $s > 0$ the functions $g_1(s), g_2(s)$ are nondecreasing; and*

$$g_1(\alpha u) \leq \psi_1(\alpha)g_1(u), \quad g_2(\alpha u) \leq \psi_2(\alpha)g_2(u)$$

*for $\alpha \geq 1, u \geq 0$, where the functions $\psi_1(\alpha), \psi_2(\alpha)$ are continuous for $\alpha \geq 1$;*

(iii) *$\int_{t_0}^\infty h(s)\mathrm{d}s = H < +\infty$.*

*Assume also that there exists a constant $K \geq 1$ such that*

$$K^{-1}H\psi_1(K)\psi_2(K) \leq \int_1^\infty \frac{\mathrm{d}s}{g_1(s^{r/p})g_2(s^{r/p})} = \frac{p}{r}\int_a^\infty \frac{\tau^{\frac{p}{r}-1}\mathrm{d}\tau}{g_1(\tau)g_2(\tau)}, \tag{2.28}$$

*where $a = (t_0)^{\frac{r}{p}}$. Then any global solution $u(t)$ of the equation (1.1) with initial data $u(t_0) = u_0, u'(t_0) = u_1$ such that $2^{p-1}(|u_0|^p + |u_1|^p) \leq K$ possesses the property (L).*

*Proof.* Without loss of generality we assume that $t_0 = 1$. With the same argument as in Theorem 2.2, for $t \geq 1$, we have

$$\begin{gathered} |u'(t)| \leq \Big[|u_1|^p + \int_1^t h(s)g_1\Big(\big[\frac{|u(s)|}{s}\big]^r\Big)g_2(|u'(s)|^r)\mathrm{d}s\Big]^{1/p}, \\ \frac{|u(t)|}{t} \leq |u_0| + \Big[|u_1|^p + \int_1^t h(s)g_1\Big(\big[\frac{|u(s)|}{s}\big]^r\Big)g_2(|u'(s)|^r)\mathrm{d}s\Big]^{1/p}. \end{gathered}$$

Applying the inequality $(A+B)^p \leq 2^{p-1}(A^p + B^p)$, $A, B \geq 0$ we obtain

$$\Big(\frac{|u(t)|}{t}\Big)^p \leq 2^{p-1}(|u_0|^p + |u_1|^p) + 2^{p-1}\Big[\int_1^t g_1\Big(\big[\frac{|u(s)|}{s}\big]^r\Big)g_2(|u'(s)|^r)\mathrm{d}s\Big]. \tag{2.29}$$

Denoting by $z(t)$ the right-hand side of inequality (2.29), for $t \geq 1$, we obtain

$$z(t) \leq K + \int_1^t H(s) g_1(z(s)^{r/p}) g_2(z(s)^{r/p}) \mathrm{d}s, \tag{2.30}$$

where $K = 2^{p-1}(|u_0|^p + |u_1|^p)$ and $H(t) = 2^{p-1}h(t)$. Assumption (ii) implies that the functions $g_1(u)$, $g_2(u)$ belong to the class $\mathbb{H}$. Furthermore, it follows from [6, Lemma 1] that if $g_1(u)$ and $g_2(u)$ belong to the class $\mathbb{H}$ with the corresponding multiplier functions $\psi_1(\alpha)$ and $\psi_2(\alpha)$ respectively, then the product $g_1(u)g_2(u)$ also belongs to $\mathbb{H}$ and the corresponding multiplier function is $\psi_1(\alpha)\psi_2(\alpha)$. Thus, applying [8, Theorem 1] to (2.30), for $t \geq 1$, we have

$$z(t) \leq K W^{-1}\Big(K^{-1}\psi_1(K)\psi_2(K) \int_1^t H(s)\mathrm{d}s\Big), \tag{2.31}$$

where

$$W(u) = \int_1^u \frac{\mathrm{d}s}{g_1(s^{r/p}) g_2(s^{r/p})}, \tag{2.32}$$

and $W^{-1}(u)$ is the inverse function for $W(u)$. Evidently, inequality (2.31) holds for all $t \geq 1$ since by (2.28)

$$K^{-1}\psi_1(K)\psi_2(K) \int_1^t H(s)\mathrm{d}s \in Dom(W^{-1}) \tag{2.33}$$

for all $t \geq 1$. The rest of the proof is analogous to that of Theorem 2.2 and is omitted. □

**Theorem 2.7.** *Let $t_0 > 0$. Suppose that assumptions (i) and (iii) of Theorem 2.6 hold, while (ii) is replaced by*

(ii') *for $s > 0$ the functions $g_1(s)$, $g_2(s)$ are continuous and nondecreasing, $g_1(0) = g_2(0) = 0$, and satisfy a Lipschitz condition*

$$|g_1(u+v) - g_1(u)| \leq \lambda_1 v, \quad |g_2(u+v) - g_2(u)| \leq \lambda_2 v,$$

*where $\lambda_1, \lambda_2$ are positive constants.*

*Then any global solution $u(t)$ of the equation (1.1) with initial data $u(t_0) = u_0$, $u'(t_0) = u_1$ such that $|u_0|^p + |u_1|^p \leq K$ possesses the property (L).*

*Proof.* Without loss of generality we may assume $t_0 = 1$. Applying [8, Corollary 2] to (2.30), we have for $t \geq 1$

$$\begin{aligned} z(t) &\leq K + g_1(K)g_2(K) \int_1^t H(s) \exp\Big(\lambda_1\lambda_2 \int_1^t H(\tau)\mathrm{d}\tau\Big)\mathrm{d}s \\ &\leq K + \bar{H} g_1(K) g_2(K) \exp(\lambda_1\lambda_2 \bar{H}) < +\infty. \end{aligned}$$

The proof of the above theorem can be completed with the same argument as in Theorem 2.2. □

**Acknowledgements.** The first author was supported by Grant No. 1/0098/08 from the Slovak Grant Agency VEGA-SAV-M. The second author was supported by Grant No. 201/08/0469 from Grant Agency of the Czech Republic.

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