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POSITIVE SOLUTIONS FOR MULTIPOINT BOUNDARY-VALUE PROBLEM WITH PARAMETERS

JUANJUAN XU, ZHONGLI WEI

ABSTRACT. In this paper, we study a generalized Sturm-Liouville boundary-value problems with two positive parameters. By constructing a completely continuous operator and combining fixed point index theorem and some properties of the eigenvalues of linear operators, we obtain sufficient conditions for the existence of at least one positive solution.

1. Introduction

Multipoint boundary-value problems for ordinary differential equations arise in different areas of applied mathematics and physics. For example, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities can be set up as a multipoint boundary-value problem; many problem in the theory of elastic stability can be handled as multipoint boundary-value problems too. Recently, the existence and multiplicity of positive solutions for nonlinear ordinary differential equations have received a great deal of attention. To identify a few cases, we refer the readers to [5, 9, 10, 11] and references therein.

Li [4] studied the following boundary-value problem (BVP for short):

$$\begin{gathered} u^{(4)}(t) + \beta u'' - \alpha u = f(t, u(t)), \quad 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{gathered} \tag{1.1}$$

where the function $f \in C([0,1] \times [0,+\infty), [0,+\infty))$, $\alpha, \beta \in \mathbb{R}$ and satisfy $\beta < 2\pi^2$, $\alpha \geq -\frac{\beta^2}{4}$, $\frac{\alpha}{\pi^4} + \frac{\beta}{\pi^2} < 1$. By applications of the fixed point index theory, sufficient conditions for existence of at least one positive solution are established.

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Ma [6] studied the existence of positive solution for BVP:

$$\begin{gathered} u^{(4)}(t)+\alpha u''-\beta u=f(t,u(t)), \quad 0<t<1, \\ u(0)=\sum_{i=1}^{m-2}\alpha_i u(\xi_i), \quad u(1)=\sum_{i=1}^{m-2}\beta_i u(\xi_i), \\ u''(0)=\sum_{i=1}^{m-2}\alpha_i u''(\xi_i), \quad u''(1)=\sum_{i=1}^{m-2}\beta_i u''(\xi_i), \end{gathered} \tag{1.2}$$

where $\alpha,\beta\in\mathbb{R}$ and $\alpha<2\pi^2$, $\beta\geq-\frac{\alpha^2}{4}$, $\alpha_i,\beta_i,\xi_i>0$ $(i=1,2,\dots,m-2)$ are constants, and $f\in C([0,1]\times[0,+\infty),[0,+\infty))$. The main tool is also the fixed point index theory.

Motivated by the results mentioned above, we are concerned with the existence of at least one positive solution for the following generalized Sturm-Liouville BVP:

$$\begin{gathered} u^{(4)}(t)-\beta u''+\alpha u=f(t,u(t)), \quad 0<t<1, \\ au(0)-bu'(0)=\sum_{i=1}^{m-2}\alpha_i u(\xi_i), \quad cu(1)+du'(1)=\sum_{i=1}^{m-2}\beta_i u(\xi_i), \\ au''(0)-bu'''(0)=\sum_{i=1}^{m-2}\alpha_i u''(\xi_i), \quad cu''(1)+du'''(1)=\sum_{i=1}^{m-2}\beta_i u''(\xi_i), \end{gathered} \tag{1.3}$$

where $f\in C([0,1]\times[0,+\infty),[0,+\infty))$ satisfying $f(t,u)\not\equiv 0$ and $\alpha,\beta\geq 0$, $a,b,c,d\in[0,+\infty)$ and $\rho:=ac+bc+ad>0$, $\xi_i\in(0,1)$, $\alpha_i,\beta_i\in[0,+\infty)$ $(i=1,2,\dots,m-2)$ are constants.

To study (1.3), we set up an integral equation which is equivalent to (1.3). By using the classical fixed point index theorem and combining some knowledge about eigenvalue of linear operator, we obtain a sufficient condition for the existence of at least one positive solution.

Following theorems are needed.

Theorem 1.1 ([3]). *Let E be a Banach space, and let $P\subset E$ be a cone. Assume $\Omega(P)$ is a bounded open set in P. Suppose that $A:\overline{\Omega(P)}\to P$ is a completely continuous operator. If there exists $\psi_0\in P\backslash\{\theta\}$ such that $\varphi-A\varphi\neq\mu\psi_0$, for all $\varphi\in\partial\Omega(P)$, $\mu\geq 0$, then the fixed point index satisfies $i(A,\Omega(P),P)=0$.*

Theorem 1.2 ([3]). *Let E be a Banach space, and let $P\subset E$ be a cone. Assume $\Omega(P)$ is a bounded open set in P with $\theta\in\Omega(P)$. Suppose that $A:\overline{\Omega(P)}\to P$ is a completely continuous operator. If $A\psi\neq\mu\psi$, for all $\psi\in\partial\Omega(P)$, $\mu\geq 1$, then the fixed point index satisfies $i(A,\Omega(P),P)=1$.*

We shall organize this paper as follows. In Section 2, we present some preliminaries and lemmas for use later. Finally, we obtain the main result and state the proof.

2. Preliminaries

In this section, we state some useful preliminary results and change the BVP (1.3) into the fixed point problem in a cone. First, we state the following hypothesis to assumed in this paper.

(H1) $\alpha, \beta \geq 0$ and $\alpha \leq \beta^2/4$.

Remark 2.1. From (H1), it follows that $\frac{\alpha}{\pi^4} + \frac{\beta}{\pi^2} > -1$.

Lemma 2.2. *Under assumption* (H1) *there exist unique* $\varphi_1, \varphi_2, \psi_1, \psi_2$ *satisfying*

$$\begin{gathered} -\varphi_i''(t) + \lambda_i \varphi_i = 0, \quad 0 < t < 1, \\ \varphi_i(0) = b, \quad \varphi_i'(0) = a, \\ -\psi_i''(t) + \lambda_i \psi_i = 0, \quad 0 < t < 1, \\ \psi_i(1) = d, \quad \psi_i'(1) = -c, \end{gathered}$$

for $i = 1, 2$. *Also on* $[0, 1]$, $\varphi_1, \varphi_2, \psi_1, \psi_2 \geq 0$, *where* λ_1, λ_2 *are the roots for the polynomial equation* $\lambda^2 - \beta\lambda + \alpha = 0$; *i.e.,*

$$\lambda_1 = \frac{\beta + \sqrt{\beta^2 - 4\alpha}}{2}, \quad \lambda_2 = \frac{\beta - \sqrt{\beta^2 - 4\alpha}}{2}.$$

Moreover, φ_1, φ_2 *are nondecreasing on* $[0, 1]$ *and* ψ_1, ψ_2 *are nonincreasing on* $[0, 1]$.

Proof. From (H1), we have $\lambda_1, \lambda_2 \geq 0$. By computations we get that: If $\lambda_i > 0$, then $\varphi_i(t) = b \cosh \sqrt{\lambda_i} t + \frac{a}{\sqrt{\lambda_i}} \sinh \sqrt{\lambda_i} t$,

$$\psi_i(t) = d \cosh \sqrt{\lambda_i}(1 - t) + \frac{c}{\sqrt{\lambda_i}} \sinh \sqrt{\lambda_i}(1 - t), \quad (i = 1, 2);$$

if $\lambda_i = 0$, then $\varphi_i(t) = b + at$, $\psi_i(t) = d + c - ct$, $(i = 1, 2)$.

It is obvious that on $[0, 1]$, $\varphi_1, \varphi_2, \psi_1, \psi_2 \geq 0$ and φ_1, φ_2 are nondecreasing on $[0, 1]$, ψ_1, ψ_2 are nonincreasing on $[0, 1]$. □

We denote

$$\rho_1 = \begin{vmatrix} \psi_1(0) & \varphi_1(0) \\ \psi_1'(0) & \varphi_1'(0) \end{vmatrix}, \quad \rho_2 = \begin{vmatrix} \psi_2(0) & \varphi_2(0) \\ \psi_2'(0) & \varphi_2'(0) \end{vmatrix},$$

$$\Delta_1 = \begin{vmatrix} -\sum_{i=1}^{m-2} \alpha_i \varphi_1(\xi_i) & \rho_1 - \sum_{i=1}^{m-2} \alpha_i \psi_1(\xi_i) \\ \rho_1 - \sum_{i=1}^{m-2} \beta_i \varphi_1(\xi_i) & -\sum_{i=1}^{m-2} \beta_i \psi_1(\xi_i) \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} -\sum_{i=1}^{m-2} \alpha_i \varphi_2(\xi_i) & \rho_2 - \sum_{i=1}^{m-2} \alpha_i \psi_2(\xi_i) \\ \rho_2 - \sum_{i=1}^{m-2} \beta_i \varphi_2(\xi_i) & -\sum_{i=1}^{m-2} \beta_i \psi_2(\xi_i) \end{vmatrix}.$$

Assume that

(H2) $\Delta_1 < 0$, $\rho_1 - \sum_{i=1}^{m-2} \alpha_i \psi_1(\xi_i) > 0$, $\rho_1 - \sum_{i=1}^{m-2} \beta_i \varphi_1(\xi_i) > 0$;

(H3) $\Delta_2 < 0$, $\rho_2 - \sum_{i=1}^{m-2} \alpha_i \psi_2(\xi_i) > 0$, $\rho_2 - \sum_{i=1}^{m-2} \beta_i \varphi_2(\xi_i) > 0$,

Similar to [8], we can get the following two lemmas by direct calculations.

Lemma 2.3. *Let* (H1)-(H2) *hold. Then for any* $g \in C[0, 1]$, *the problem*

$$\begin{gathered} -u''(t) + \lambda_1 u(t) = g(t), \quad 0 < t < 1, \\ au(0) - bu'(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad cu(1) + du'(1) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \end{gathered} \tag{2.1}$$

has a unique solution $u(t) = \int_0^1 G_1(t, s) g(s)\, ds + A_1(g)\varphi_1(t) + B_1(g)\psi_1(t)$, *where*

$$G_1(t, s) = \frac{1}{\rho_1} \begin{cases} \varphi_1(t)\psi_1(s), & 0 \leq t \leq s \leq 1, \\ \varphi_1(s)\psi_1(t), & 0 \leq s \leq t \leq 1, \end{cases}$$

$$A_1(g) := \frac{1}{\Delta_1}\begin{vmatrix} \sum_{i=1}^{m-2}\alpha_i\int_0^1 G_1(\xi_i,s)g(s)\,ds & \rho_1-\sum_{i=1}^{m-2}\alpha_i\psi_1(\xi_i) \\ \sum_{i=1}^{m-2}\beta_i\int_0^1 G_1(\xi_i,s)g(s)\,ds & -\sum_{i=1}^{m-2}\beta_i\psi_1(\xi_i)\end{vmatrix},$$

$$B_1(g) := \frac{1}{\Delta_1}\begin{vmatrix} -\sum_{i=1}^{m-2}\alpha_i\varphi_1(\xi_i) & \sum_{i=1}^{m-2}\alpha_i\int_0^1 G_1(\xi_i,s)g(s)\,ds \\ \rho_1-\sum_{i=1}^{m-2}\beta_i\varphi_1(\xi_i) & \sum_{i=1}^{m-2}\beta_i\int_0^1 G_1(\xi_i,s)g(s)\,ds\end{vmatrix},$$

and where $g \ge 0$, $u(t) \ge 0$, $t \in [0,1]$.

The proof of the above lemma follows by routine calculations.

Lemma 2.4. *Let* (H1), (H3) *hold. Then for each* $g \in C[0,1]$, *the problem*

$$\begin{gathered} -u''(t)+\lambda_2 u(t) = g(t), \quad 0<t<1, \\ au(0)-bu'(0) = \sum_{i=1}^{m-2}\alpha_i u(\xi_i), \quad cu(1)+du'(1) = \sum_{i=1}^{m-2}\beta_i u(\xi_i), \end{gathered} \tag{2.2}$$

has a unique solution $u(t) = \int_0^1 G_2(t,s)g(s)\,ds + A_2(g)\varphi_2(t) + B_2(g)\psi_2(t)$, *where*

$$G_2(t,s) = \frac{1}{\rho_2}\begin{cases}\varphi_2(t)\psi_2(s), & 0\le t\le s\le 1,\\ \varphi_2(s)\psi_2(t), & 0\le s\le t\le 1,\end{cases}$$

$$A_2(g) := \frac{1}{\Delta_2}\begin{vmatrix} \sum_{i=1}^{m-2}\alpha_i\int_0^1 G_2(\xi_i,s)g(s)\,ds & \rho_2-\sum_{i=1}^{m-2}\alpha_i\psi_2(\xi_i) \\ \sum_{i=1}^{m-2}\beta_i\int_0^1 G_2(\xi_i,s)g(s)\,ds & -\sum_{i=1}^{m-2}\beta_i\psi_2(\xi_i)\end{vmatrix},$$

$$B_2(g) := \frac{1}{\Delta_2}\begin{vmatrix} -\sum_{i=1}^{m-2}\alpha_i\varphi_2(\xi_i) & \sum_{i=1}^{m-2}\alpha_i\int_0^1 G_2(\xi_i,s)g(s)\,ds \\ \rho_2-\sum_{i=1}^{m-2}\beta_i\varphi_2(\xi_i) & \sum_{i=1}^{m-2}\beta_i\int_0^1 G_2(\xi_i,s)g(s)\,ds\end{vmatrix},$$

and $g \ge 0$, $u(t) \ge 0$, $t \in [0,1]$.

The proof of the above lemma follows by routine calculations.

Remark 2.5. Suppose that (H2) and (H3) hold. It follows that $A_i(g), B_i(g)$ ($i = 1,2$) are increasing.

Lemma 2.6. *Assume that* (H1)–(H3) *hold. Then* (1.3) *has a unique solution*

$$\begin{aligned} u(t) &= \int_0^1\int_0^1 G_2(t,\tau)G_1(\tau,s)f(s,u(s))\,ds\,d\tau + \int_0^1 G_2(t,\tau)A_1(f)\varphi_1(\tau)\,d\tau \\ &\quad + \int_0^1 G_2(t,\tau)B_1(f)\psi_1(\tau)\,d\tau + A_2(h)\varphi_2(t) + B_2(h)\psi_2(t), \end{aligned} \tag{2.3}$$

where $G_1, G_2, A_1, A_2, B_1, B_2$ *are defined as above,*

$$h(t) = \int_0^1 G_1(t,s)f(s,u(s))\,ds + A_1(f)\varphi_1(t) + B_1(f)\psi_1(t).$$

Obviously, $u(t) \ge 0$ for all $t \in [0,1]$. Let $E = C[0,1]$ and $P = \{u \in E, u \ge 0\}$. It is obvious that P is a cone in E. Define $T : E \to E$,

$$\begin{aligned} Tu(t) &= \int_0^1\int_0^1 G_2(t,\tau)G_1(\tau,s)f(s,u(s))\,ds\,d\tau + \int_0^1 G_2(t,\tau)A_1(f)\varphi_1(\tau)\,d\tau \\ &\quad + \int_0^1 G_2(t,\tau)B_1(f)\psi_1(\tau)\,d\tau + A_2(h)\varphi_2(t) + B_2(h)\psi_2(t), \end{aligned} \tag{2.4}$$

where $h(t) = \int_0^1 G_1(t,s)f(s,u(s))\,ds + A_1(f)\varphi_1(t) + B_1(f)\psi_1(t)$.

We can easily obtain that u is a positive solution of (1.3) if and only if u is a fixed point of T in P.

Define $L : E \to E$,

$$\begin{aligned} Lu(t) = &\int_0^1 \int_0^1 G_2(t,\tau)G_1(\tau,s)u(s)\,ds\,d\tau + \int_0^1 G_2(t,\tau)A_1(u)\varphi_1(\tau)\,d\tau \\ &+ \int_0^1 G_2(t,\tau)B_1(u)\psi_1(\tau)\,d\tau + A_2(e)\varphi_2(t) + B_2(e)\psi_2(t), \end{aligned} \tag{2.5}$$

where $e(t) = \int_0^1 G_1(t,s)u(s)\,ds + A_1(u)\varphi_1(t) + B_1(u)\psi_1(t)$.

Lemma 2.7. *Suppose that* (H1)–(H3) *hold. Then $T : P \to P$ is completely continuous. Also $L : P \to P$ is completely continuous.*

Lemma 2.8. *Suppose that* (H1)–(H3) *hold. Then for the operator L defined by* (2.5)*, the spectral radius $r(L) \neq 0$ and L has a positive eigenfunction corresponding to its first eigenvalue $\lambda_* = r(L)^{-1}$.*

Proof. It is easy to see that there is $t_1 \in (0,1)$, such that $G_1(t_1,t_1)G_2(t_1,t_1) > 0$. Thus there exists $[\alpha,\beta] \subset (0,1)$ such that $t_1 \in (\alpha,\beta)$ and $G_1(t,\tau)G_2(\tau,s) > 0$, $t,\tau,s \in [\alpha,\beta]$.

Take $u \in E$ such that $u(t) \geq 0$ for all $t \in [0,1]$, $u(t_1) > 0$ and $u(t) = 0$ for all $t \in [0,1]\backslash[\alpha,\beta]$. Then for $t \in [\alpha,\beta]$,

$$\begin{aligned} Lu(t) &= \int_0^1 \int_0^1 G_2(t,\tau)G_1(\tau,s)u(s)\,ds\,d\tau + \int_0^1 G_2(t,\tau)A_1(u)\varphi_1(\tau)\,d\tau \\ &\quad + \int_0^1 G_2(t,\tau)B_1(u)\psi_1(\tau)\,d\tau + A_2(e)\varphi_2(t) + B_2(e)\psi_2(t) \\ &\geq \int_\alpha^\beta \int_\alpha^\beta G_2(t,\tau)G_1(\tau,s)u(s)\,ds\,d\tau + \int_\alpha^\beta G_2(t,\tau)A_1(u)\varphi_1(\tau)\,d\tau \\ &\quad + \int_\alpha^\beta G_2(t,\tau)B_1(u)\psi_1(\tau)\,d\tau + A_2(e)\varphi_2(t) + B_2(e)\psi_2(t) > 0. \end{aligned}$$

So there exists a constant $c > 0$ such that for $t \in [0,1]$, $c(Lu)(t) \geq u(t)$. From Krein-Rutmann Theorem [3], we know that the spectral radius $r(L) \neq 0$ and L has a positive eigenfunction corresponding to its first eigenvalue $\lambda_* = r(L)^{-1}$. □

3. Main Result

Theorem 3.1. *Suppose that* (H1)–(H3) *hold, and $\underline{f_0} > \lambda_*, \overline{f_\infty} < \lambda_*$, where λ_* is the first eigenvalue of L defined by* (2.5)*. Then* (1.3) *has at least one positive solution, where*

$$\underline{f_0} = \liminf_{u\to 0^+} \min_{t\in[0,1]} \frac{f(t,u)}{u}, \quad \overline{f_\infty} = \limsup_{u\to+\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u}.$$

Proof. From $\underline{f_0} > \lambda_*$, there exists $r_1 > 0$, such that $f(t,u) \geq \lambda_* u$ for all $t \in [0,1]$, $u \in [0,r_1]$. Let $u \in \partial B_{r_1} \cap P$. Then

$$\begin{aligned} Tu(t) = &\int_0^1 \int_0^1 G_2(t,\tau)G_1(\tau,s)f(s,u(s))\,ds\,d\tau + \int_0^1 G_2(t,\tau)A_1(f)\varphi_1(\tau)\,d\tau \\ &+ \int_0^1 G_2(t,\tau)B_1(f)\psi_1(\tau)\,d\tau + A_2(h)\varphi_2(t) + B_2(h)\psi_2(t) \end{aligned}$$

$$\begin{aligned}
&\ge \lambda_*[\int_0^1\int_0^1 G_2(t,\tau)G_1(\tau,s)u(s)\,ds\,d\tau + \int_0^1 G_2(t,\tau)A_1(u)\varphi_1(\tau)\,d\tau \\
&\quad + \int_0^1 G_2(t,\tau)B_1(u)\psi_1(\tau)\,d\tau + A_2(e)\varphi_2(t) + B_2(e)\psi_2(t)] \\
&= \lambda_*(Lu)(t).
\end{aligned}$$

We may suppose that T has no fixed point on $\partial B_{r_1}\cap P$(otherwise, the proof is complete). Now we show that $u - Tu \ne \mu u^*$ for all $u \in \partial B_{r_1}\cap P$, $\mu \ge 0$.

Otherwise, there exists $u_1 \in \partial B_{r_1}\cap P$, $\tau_0 \ge 0$, such that $u_1 - Tu_1 = \tau_0 u^*$, that is

$$u_1 = Tu_1 + \tau_0 u^*.$$

Let $\tau^* = \sup\{\tau : u_1 \ge \tau u^*\}$, then $\tau^* \ge \tau_0 > 0$, and $u_1 \ge \tau^* u^*$. Since $L(P) \subset P$, $\lambda_* Lu_1 \ge \tau^*\lambda_* Lu^* = \tau^* u^*$, we have

$$u_1 = Tu_1 + \tau_0 u^* \ge \lambda_* Lu_1 + \tau_0 u^* \ge (\tau^* + \tau_0)u^*.$$

which contradicts the definition of τ^*, so $i(T, B_{r_1}\cap P, P) = 0$.

From $\overline{f_\infty} < \lambda_*$, there exits $0 < \sigma < 1$, $r_2 > r_1$, such that $f(t,u) \le \sigma\lambda_* u$ for all $t \in [0,1]$, $u \in [r_2, +\infty)$. Let $L_1 u = \sigma\lambda_* Lu$, $u \in E$, then $L_1 : E \to E$ is a bounded linear operator and $L_1(P) \subset P$. Let

$$\begin{aligned}
M^* = \max_{u\in \overline{B}_{r_2}\cap P, t\in[0,1]} &\int_0^1\int_0^1 G_2(t,\tau)G_1(\tau,s)f(s,u(s))\,ds\,d\tau \\
&+ \int_0^1 G_2(t,\tau)A_1(f)\varphi_1(\tau)\,d\tau + \int_0^1 G_2(t,\tau)B_1(f)\psi_1(\tau)\,d\tau \\
&+ A_2(h)\varphi_2(t) + B_2(h)\psi_2(t),
\end{aligned}$$

obviously, $0 < M^* < +\infty$. Let $W = \{u \in P : u = \mu Tu,\ 0 \le \mu \le 1\}$, for all $u \in W$, denote $\widehat{u(t)} = \min\{u(t), r_2\},\ s(u) = \{t \in [0,1], u(t) > r_2\},\ \widehat{f(t)} = f(t, \widehat{u(t)})$. Then

$$\begin{aligned}
u(t) &= \mu Tu(t) \le Tu(t) \\
&= \int_0^1\int_{s(u)} G_2(t,\tau)G_1(\tau,s)f(s,u(s))\,ds\,d\tau + \int_0^1 G_2(t,\tau)A_{1\,s(u)}(f)\varphi_1(\tau)\,d\tau \\
&\quad + \int_0^1 G_2(t,\tau)B_{1\,s(u)}(f)\psi_1(\tau)\,d\tau + A_2(h_{s(u)})\varphi_2(t) + B_2(h_{s(u)})\psi_2(t) \\
&\quad + \int_0^1\int_{[0,1]/s(u)} G_2(t,\tau)G_1(\tau,s)f(s,u(s))\,ds\,d\tau \\
&\quad + \int_0^1 G_2(t,\tau)A_{1\,[0,1]/s(u)}(f)\varphi_1(\tau)\,d\tau \\
&\quad + \int_0^1 G_2(t,\tau)B_{1\,[0,1]/s(u)}(f)\psi_1(\tau)\,d\tau \\
&\quad + A_2(h_{[0,1]/s(u)})\varphi_2(t) + B_2(h_{[0,1]/s(u)})\psi_2(t) \\
&\le \sigma\lambda_*[\int_0^1\int_0^1 G_2(t,\tau)G_1(\tau,s)u(s)\,ds\,d\tau + \int_0^1 G_2(t,\tau)A_1(u)\varphi_1(\tau)\,d\tau \\
&\quad + \int_0^1 G_2(t,\tau)B_1(u)\psi_1(\tau)\,d\tau + A_2(e)\varphi_2(t) + B_2(e)\psi_2(t)]
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1\int_0^1 G_2(t,\tau)G_1(\tau,s)f(s,\widehat{u(s)})\,ds\,d\tau+\int_0^1 G_2(t,\tau)A_1(\widehat{f})\varphi_1(\tau)\,d\tau\\
&+\int_0^1 G_2(t,\tau)B_1(\widehat{f})\psi_1(\tau)\,d\tau+A_2(\widehat{f})\varphi_2(t)+B_2(\widehat{f})\psi_2(t)\\
&\le (L_1u)(t)+M^*,\quad t\in[0,1].
\end{aligned}
$$

where

$$
A_{1s(u)}(f):=\frac{1}{\Delta_1}\begin{vmatrix}\sum_{i=1}^{m-2}\alpha_i\int_{s(u)}G_1(\xi_i,s)g(s)\,ds & \rho_1-\sum_{i=1}^{m-2}\alpha_i\psi_1(\xi_i)\\ \sum_{i=1}^{m-2}\beta_i\int_{s(u)}G_1(\xi_i,s)g(s)\,ds & -\sum_{i=1}^{m-2}\beta_i\psi_1(\xi_i)\end{vmatrix},
$$

$B_{1s(u)}$, $A_{2[0,1]/s(u)}$, $B_{2[0,1]/s(u)}$ have the similar meaning and

$$
h_{s(u)}(t)=\int_{s(u)}G_1(t,s)f(s,u(s))\,ds+A_{1s(u)}(f)\varphi_1(t)+B_{1s(u)}(f)\varphi_2(t).
$$

Thus

$$
(I-L_1)u\le M^*,\quad t\in[0,1].
$$

Since $u^*=\lambda_*(Lu^*)$ and $0<\sigma<1$, we have $r(L_1)^{-1}>1$; i.e., $(I-L_1)^{-1}$ exists and

$$
(I-L_1)^{-1}=I+L_1+L_1^2+\dots+L_1^n+\dots.
$$

It follows from $L_1(P)\subset P$ that $(I-L_1)^{-1}(P)\subset P$. Therefore, $u(t)\le(I-L_1)^{-1}M^*$, $t\in[0,1]$, and W is bounded. We denote by $\sup W$ the bound of W.

Select $r_3>\max\{r_2,\sup W\}$, then for all $u\in\partial B_{r_3}\cap P$, $u\ne\mu Tu$, $0\le\mu\le1$; that is,

$$
Tu\ne\frac{1}{\mu}u,\quad \frac{1}{\mu}\ge1,\quad \forall u\in\partial B_{r_3}\cap P,
$$

so from Theorem 1.2, we have $i(T,B_{r_3}\cap P,P)=1$. Therefore,

$$
i(T,(B_{r_3}\cap P)\backslash(\overline{B}_{r_1}\cap P),P)=i(T,B_{r_3}\cap P,P)-i(T,B_{r_1}\cap P,P)=1.
$$

By the solution properties of the fixed point index, T has at least one fixed point on $(B_{r_3}\cap P)\backslash(\overline{B}_{r_1}\cap P)$, which means that the generalized Sturm-Liouville boundary-value problem (1.3) has at least one positive solution. □

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References

[1] Chai, Guoqing; *Existence of positive solutions for second-order boundary-value problem with one parameter*, J. Math. Anal. Appl., 330 (2007): 541-549.

[2] Guo, Dajun; Sun, Jingxian; *Nonlinear Integral Equations*, Shandong Science and Technology Press, Jinan, 1987. (in Chinese)

[3] Guo, D., Lakshmikantham,v.; *Nonlinear Problems in Abstract Cones*, Academic Press Inc., New York, 1988.

[4] Li, Yongxiang; *Positive solutions of fourth-order boundary-value problems with two parameters*, J. Math. Anal. Appl., 281 (2003): 477-484.

[5] Ma, Huili; *Symmetric positive solutions for nonlacal boundary-value problem of fourth order*, Nonlinear Anal., 68(2008): 645-651.

[6] Ma, Huili; *Positive solution for m-point boundary-value problem of fourth-order*, J. Math. Anal. Appl., 321(2006): 37-49.

[7] Ma, Ruyun; *Nonlocal problems in nonlinear ordinary differential problems*, Science press, Beijing, 2004. (in Chinese)

[8] Ma, Ruyun, Thompson, Bevan; *Positive solutions for nonlinear m-point eigenvalue problems*, J. Math. Anal. Appl., 297(2004): 24-37.

[9] Zhang, Xuemei; Feng, Meiqiang; Ge, Weigao; *Multiple positive solutions for a class of m-point boundary-value problems*, Appl. Math. Lett. (2007), doi:10.1016/ j.aml.2007.10.019.

[10] Zhang, Xinguang; Liu, Lishan; *Eigenvalue of fourth-order m-point boundary-value problems with two deviatives*, Comp. Math. Appl. (2008) doi:10.1016/ j.camwa.2007.08.048.

[11] Zhang, Mingchuan; Wei, Zhongli; *Existence of positive solutions for fourth-order m-point boundary-value problems with variable parameters*, Appl. Math. Comp., 190 (2007): 1417-1431.

Juanjuan Xu
School of Mathematics, Shandong University, Jinan, Shandong 250100, China
E-mail address: `jnxujuanjuan@163.com Tel: 86-531-88369649`

Zhongli Wei
School of Mathematics, Shandong University, Jinan, Shandong 250100, China
E-mail address: `jnwzl@yahoo.com.cn`