

WELL-POSEDNESS AND ILL-POSEDNESS OF THE FIFTH-ORDER MODIFIED KdV EQUATION

SOONSIK KWON

ABSTRACT. We consider the initial value problem of the fifth-order modified KdV equation on the Sobolev spaces.

$$\begin{aligned}\partial_t u - \partial_x^5 u + c_1 \partial_x^3(u^3) + c_2 u \partial_x u \partial_x^2 u + c_3 u u \partial_x^3 u &= 0 \\ u(x, 0) &= u_0(x)\end{aligned}$$

where $u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and c_j 's are real. We show the local well-posedness in $H^s(\mathbb{R})$ for $s \geq 3/4$ via the contraction principle on $X^{s,b}$ space. Also, we show that the solution map from data to the solutions fails to be uniformly continuous below $H^{3/4}(\mathbb{R})$. The counter example is obtained by approximating the fifth order mKdV equation by the cubic NLS equation.

1. INTRODUCTION

The KdV equation and the modified KdV (mKdV) equation are completely integrable in the sense that there are Lax pair formulations. Being completely integrable, the KdV and the mKdV equations enjoy infinite number of conservation laws. Each of these is an Hamiltonian of the flow which commute the KdV flow (resp. the mKdV flow). This generates an infinite collection of commuting nonlinear equations of order $2j+1$, ($j \in \mathbb{N}$), which is known as the KdV hierarchy (resp. the mKdV hierarchy). In this note, we consider the second equation from the modified KdV hierarchy:

$$\partial_t u - \partial_x^5 u - 30u^4 \partial_x u + 10u^2 \partial_x^3 u + 10(\partial_x u)^3 + 40u \partial_x u \partial_x^2 u = 0. \quad (1.1)$$

Using the theory of the complete integrability, one can show that for any Schwartz initial data, the solution to any equation in the KdV hierarchy (resp. the mKdV hierarchy) exists globally in time. However, the well-posedness theory for low regularity initial data requires the theory of dispersive PDEs. And changing coefficients in the nonlinear terms may break the integrable structure. In this case, we can no longer rely on the theory of complete integrability. The purpose of this paper is to study the low regularity well-posedness and ill-posedness. We consider the

2000 *Mathematics Subject Classification.* 35Q53.

Key words and phrases. Local well-posedness; ill-posedness; mKdV hierarchy.

©2008 Texas State University - San Marcos.

Submitted November 7, 2007. Published January 2, 2008.

following fifth order mKdV equation, which generalizes (1.1) ¹.

$$\begin{aligned} \partial_t u - \partial_x^5 u + c_1 \partial_x^3 (u^3) + c_2 u \partial_x u \partial_x^2 u + c_3 u u \partial_x^3 u &= 0 \\ u(x, 0) &= u_0(x) \end{aligned} \quad (1.2)$$

where $u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and c_j 's are real.

We show the local well-posedness result and the ill-posedness result. First, we state the local well-posedness theorem.

Theorem 1.1. *Let $s \geq 3/4$ and $u_0 \in H^s(\mathbb{R})$. Then there exists $T = T(\|u_0\|_{H^s(\mathbb{R})})$ such that (1.2) has a unique solution $u(t, x)$ in $C([0, T]; H^s(\mathbb{R}))$. Moreover, the solution map from data to the solutions is real-analytic.*

Previously, Kenig, Ponce, and Vega [6] studied the local well-posedness of the odd order dispersive equations:

$$\partial_t u + \partial_x^{2j+1} u + P(u, \partial_x u, \dots, \partial_x^{2j} u) = 0$$

where P is a polynomial having no constant and linear terms. They proved the local well-posedness for the initial data in the weighted Sobolev space; i.e.,

$$u_0 \in H^s(\mathbb{R}) \cap L^2(|x|^m dx)$$

for some $s, m \geq 0$. Their method was the iteration using the local smoothing estimate and the maximal function estimate. Inspecting their proof for the equation (1.2), one can observe that the local well-posedness holds true for $s > 9/4$ and $m = 0$. In other words, the local well-posedness is established for the Sobolev space *without the decaying weight*. Thus, our result can be viewed as an improvement of theirs. Our proof is also via the contraction principle. A natural choice of the iteration space is the Bourgain space, also known as the $X^{s,b}$ space. Assuming the standard argument of the iteration on the $X^{s,b}$ space, the main step is to show the following nonlinear estimate:

$$\|T(u, v, w)\|_{X^{s,b-1}} \lesssim \|u\|_{X^{s,b}} \|v\|_{X^{s,b}} \|w\|_{X^{s,b}}.$$

where $T(u, v, w) = c_1 \partial_x^3 (uvw) + c_2 u \partial_x v \partial_x^2 w + c_3 uv \partial_x^3 w$. This is performed by the dyadic method of Tao. In [10], Tao studied multilinear estimates for $X^{s,b}$ space systematically and showed the analogous trilinear estimate for the mKdV equation. This reproves the local well-posedness for $s \geq 1/4$, which originally showed by Kenig, Ponce and Vega [5] by the local smoothing estimate. Thus, in the mKdV equation the $X^{s,b}$ estimate has the same strength as the classical local smoothing method, while in the fifth order mKdV (1.2) the $X^{s,b}$ estimate improves the preceding one.

In [3], Christ, Colliander and Tao showed the solution map of the mKdV equation fails to be uniformly continuous for $s < 1/4$. This implies $1/4$ is the minimal regularity threshold for which the well-posedness problem can be solved via an iteration method. Our next theorem is the analogue of this for the equation (1.2).

Theorem 1.2. *Let $-7/24 < s < 3/4$. The solution map of the initial value problem (1.2) fails to be uniformly continuous. More precisely, for $0 < \delta \ll \epsilon \ll 1$ and $T > 0$*

¹For omitting $u^4 \partial_x u$, See Remark 3.5

arbitrary, there are two solutions u, v to (1.2) such that

$$\|u(0)\|_{H_x^s}, \|v(0)\|_{H_x^s} \lesssim \epsilon \tag{1.3}$$

$$\|u(0) - v(0)\|_{H_x^s} \lesssim \delta \tag{1.4}$$

$$\sup_{0 \leq t \leq T} \|u(t) - v(t)\|_{H_x^s} \gtrsim \epsilon. \tag{1.5}$$

The method used here is very similar to theirs in [3]. We approximate the fifth order mKdV equation by the cubic NLS equation,

$$i\partial_t u - \partial_x^2 u + |u|^2 u = 0, \tag{1.6}$$

at (N, N^5) in the frequency space.

Let $u(t, x)$ be the linear solution to $(\partial_t - \partial_x^2)u = 0$ with $u(0) = u_0$. Setting

$$\xi := N + \frac{\xi'}{\sqrt{10}N^{3/2}},$$

$\tau = \xi^5$ leads $\tau = N^5 + \sqrt{\frac{5}{2}}N^{5/2}\xi' + \tau'$ where

$$\tau' = \xi'^2 + \frac{\xi'^3}{\sqrt{10}N^{5/2}} + \frac{\xi'^4}{20N^5} + \frac{\xi'^5}{(10N^3)^{5/2}}.$$

$$\begin{aligned} u(t, x) &= \int e^{it\tau + ix\xi} \widehat{u_0}(\xi) d\tau d\xi \\ &= \int e^{it(N^5 + \sqrt{\frac{5}{2}}N^{5/2}\xi' + \tau') + ix(N + \frac{\xi'}{\sqrt{10}N^{3/2}})} \widehat{u_0}(\xi) d\tau d\xi \\ &= e^{iN^5 + iNx} \int e^{i\tau' t + i\xi'(\frac{x}{\sqrt{10}N^{3/2}} + \sqrt{\frac{5}{2}}N^{5/2}t)} \widehat{u_0}(N + \frac{\xi'}{\sqrt{10}N^{3/2}}) d\tau' d\xi'. \end{aligned}$$

Since $\tau' \approx \xi'^2$ for $|\xi'| \ll N$,

$$u(t, x) \approx e^{iN^5 t + iNx} v(t, \frac{x}{\sqrt{10}N^{3/2}} + \sqrt{\frac{5}{2}}N^{5/2}t)$$

where $v(t, x)$ is a solution to the linear Schrödinger equation $i\partial_t v - \partial_x^2 v = 0$.

By the presence of the nonlinear term, one need the factor $\frac{c}{N^{3/2}}$ and the real part projection Re . Then it is approximated to the cubic NLS equation (1.6). The detail follows in Section 4.

On the other hand, the solutions to the fifth order KdV equation,

$$\partial_t u - \partial_x^5 u + c_1 \partial_x u \partial_x^2 u + c_2 u \partial_x^3 u = 0,$$

is known to have genuine nonlinear dynamics for all $s > 0$. In [9] the author showed the solution map fails to be uniformly continuous in $H^s(\mathbb{R})$ for $s > 0$. Thus, for this equation the local well-posedness problem is solved by other than the iteration method. In [9] the local well-posedness in $H^s(\mathbb{R})$ for $s > \frac{5}{2}$ is established via the compactness method.

Notation. We use $X \lesssim Y$ when $X \leq CY$ for some C . We use $X \sim Y$ when $X \lesssim Y$ and $Y \lesssim X$. Moreover, we use $X \lesssim_s Y$ if the implicit constant depends on s , $C = C(s)$.

We use Japanese bracket notation $\langle \xi \rangle := \sqrt{1 + \xi^2}$. We denote the space time Fourier transform by $\tilde{u}(\tau, \xi)$ of $u(t, x)$

$$\tilde{u}(\tau, \xi) = \int e^{-it\tau - ix\xi} u(t, x) dt dx,$$

while the space Fourier transform by $\hat{u}(t, \xi)$ of $u(t, x)$,

$$\hat{u}(t, \xi) = \int e^{-ix\xi} u(t, x) dx.$$

2. LOCAL WELL-POSEDNESS OF THE FIFTH ORDER MODIFIED KDV

In this section, we prove the local well-posedness of the initial value problem (1.2). Our proof is via the contraction principle on the Bourgain space. We first recall some standard facts and notations. For a Schwartz function $u_0(x)$, we denote the linear solution $u(t, x)$ to the equation $\partial_t u - \partial_x^5 u = 0$ by

$$u(t, x) =: e^{t\partial_x^5} u_0(x) = c \iint e^{it\xi^5} e^{i(x-y)\xi} u_0(y) dy d\xi.$$

Using this notation we have the Duhamel formula for the solution to the inhomogeneous linear equation $\partial_t u - \partial_x^5 u + F = 0$

$$u(t, x) = e^{t\partial_x^5} u_0(x) - \int_0^t e^{(t-t')\partial_x^5} F(t', x) dt'.$$

We denote the Bourgain space by $X_{\tau=\xi^5}^{s,b}(\mathbb{R} \times \mathbb{R})$, or abbreviated $X^{s,b}$. The $X^{s,b}$ space is defined to be the closure of the Schwartz functions $\mathcal{S}(\mathbb{R} \times \mathbb{R})$ under the norm

$$\|u\|_{X_{\tau=\xi^5}^{s,b}(\mathbb{R} \times \mathbb{R})} := \|\langle \xi \rangle^s \langle \tau - \xi^5 \rangle^b \tilde{u}(\tau, \xi)\|_{L_{\tau,\xi}^2(\mathbb{R} \times \mathbb{R})}.$$

The $X^{s,b}$ space is continuously embedded in $C_t^0 H_x^s$.

Lemma 2.1. *Let $b > 1/2$ and $s \in \mathbb{R}$. Then for any $u \in X_{\tau=\xi^5}^{s,b}(\mathbb{R} \times \mathbb{R})$, we have*

$$\|u\|_{C_t^0 H_x^s(\mathbb{R} \times \mathbb{R})} \lesssim_b \|u\|_{X_{\tau=\xi^5}^{s,b}(\mathbb{R} \times \mathbb{R})}.$$

For the proof of the above lemma, see [11]. Let $\eta(t)$ be a compactly supported smooth time cut-off function (i.e. $\eta \in C_0^\infty(\mathbb{R})$ with $\eta(t) = 1$ on $[0, 1]$). There is a standard $X^{s,b}$ energy estimate for time cut-off solutions.

Lemma 2.2. *Let $b > 1/2$ and $s \in \mathbb{R}$ and let $u \in C_t^\infty \mathcal{S}_x(\mathbb{R} \times \mathbb{R})$ solves the inhomogeneous linear fifth order KdV equation $\partial_t u - \partial_x^5 u = F$. Then we have*

$$\|\eta(t)u\|_{X_{\tau=\xi^5}^{s,b}(\mathbb{R} \times \mathbb{R})} \lesssim_{\eta,b} \|u(0)\|_{H_x^s} + \|\eta(t)F\|_{X_{\tau=\xi^5}^{s,b-1}(\mathbb{R} \times \mathbb{R})}. \quad (2.1)$$

For the proof of the above Lemma, see [7], [11]. Next, we state the nonlinear estimate.

Proposition 2.3. *Let $s \geq 3/4$. For all u, v, w on $\mathbb{R} \times \mathbb{R}$ and $1/2 < b \leq 1/2 + \epsilon$ for some ϵ , we have*

$$\begin{aligned} & \|\partial_x^3(uvw)\|_{X^{s,b-1}} + \|uv\partial_x^3 w\|_{X^{s,b-1}} + \|u\partial_x^2 v\partial_x w\|_{X^{s,b-1}} + \|\partial_x u\partial_x v\partial_x w\|_{X^{s,b-1}} \\ & \lesssim \|u\|_{X^{s,b}} \|v\|_{X^{s,b}} \|w\|_{X^{s,b}}. \end{aligned} \quad (2.2)$$

Combining the preceding estimates (2.1), (2.2) one can easily verify that the operator

$$\Phi(u)(t, x) := \eta(t)e^{t\partial_x^5}u_0(x) - \eta(t) \int_0^t e^{(t-t')\partial_x^5}F(u)(t', x)dt'$$

is a contraction on a ball of $X^{s,b}$ space

$$\mathcal{B} = \{u \in X^{s,b} : \|u\|_{X^{s,b}} \leq 2\delta\}$$

for a sufficiently small $\delta > 0$ and $\|u_0\|_{H_x^s} < \delta$, where $F(u) = c_1\partial_x^3(u^3) + c_2u\partial_xu\partial_x^2u + c_3uu\partial_x^3u$. This proves the local well-posedness for small data. Then a standard scaling argument easily leads the local well-posedness for large data. Once the local well-posedness is proved via the contraction principle, we also obtain that the solution map is Lipschitz continuous, and furthermore if the nonlinear term is algebraic (a polynomial of u and its derivatives), then the solution map is real-analytic. Hence, it remains to show the trilinear estimate (2.2) for the proof of Theorem 1.1.

3. TRILINEAR ESTIMATE

In this section, we show the trilinear estimate (2.2). We closely follow the method developed by Tao [10] in the context of modified KdV equation. Writing the trilinear inequality in the dual form and we view it as a composition of two bilinear operators based on L^2 norm. Then we reduce to two bilinear estimates. First, we recall notations and general frame work of Tao's $[k; Z]$ -multiplier method. For the details we refer to [10].

Notation and block estimates. We define $[k, \mathbb{R}]$ -multiplier norm of Tao [10] first. Let Z be an abelian additive group with an invariant measure $d\xi$ (for instance $\mathbb{R}^n, \mathbb{T}^n$). For any integer $k \geq 2$. let $\Gamma_k(Z)$ denote the hyperplane

$$\Gamma_k(Z) := \{(\xi_1, \dots, \xi_k) \in \mathbb{R}^k : \xi_1 + \dots + \xi_k = 0\}.$$

A $[k, Z]$ -multiplier is defined to be any function $m : \Gamma_k(Z) \rightarrow \mathbb{C}$. Then we define the multiplier norm $\|m\|_{[k, Z]}$ to be the best constant so that the inequality

$$\left| \int_{\Gamma_k(Z)} m(\xi) \prod_{j=1}^k f_j(\xi_j) \right| \leq C \prod_{j=1}^k \|f_j\|_{L^2},$$

holds for all functions f_j on Z . Any capitalized variables such as N_j, L_j and H are presumed to be dyadic. For $N_1, N_2, N_3 > 0$, we denote the quantities by $N_{\min}, N_{\text{med}}, N_{\max}$ in their order and similarly for L_1, L_2, L_3 . We adopt the following summation convention:

$$\begin{aligned} \sum_{N_{\max} \sim N_{\text{med}} \sim N} & := \sum_{\substack{N_1, N_2, N_3 > 0 \\ N_{\max} \sim N_{\text{med}} \sim N}}, \\ \sum_{L_{\max} \sim H} & := \sum_{\substack{L_1, L_2, L_3 \gtrsim 1 \\ L_{\max} \sim H}}. \end{aligned}$$

For given τ_j, ξ_j with $\xi_1 + \xi_2 + \xi_3 = 0$ and $\tau_1 + \tau_2 + \tau_3 = 0$, we denote the modulation

$$\tau_j - \xi_j^5 =: \lambda_j$$

and the resonance function

$$h(\xi) := \xi_1^5 + \xi_2^5 + \xi_3^5 = -\lambda_1 - \lambda_2 - \lambda_3.$$

By a dyadic decomposition of the variables ξ_j, λ_j and $h(\xi) X^{s,b}$, a bilinear estimate

$$\|B(u, v)\|_{X^{s_3, b_3}} \lesssim \|u\|_{X^{s_1, b_1}} \|v\|_{X^{s_2, b_2}}$$

is reduced to

$$\begin{aligned} & \left\| \sum_{N_{\max} \gtrsim 1} \sum_H \sum_{L_1, L_2, L_3 \gtrsim 1} \frac{\tilde{m}(N_1, N_2) \langle N_1 \rangle^{-s_1} \langle N_2 \rangle^{-s_2} \langle N_3 \rangle^{s_3}}{L_1^{b_1} L_2^{b_2} L_3^{-b_3}} \right. \\ & \left. \times X_{N_1, N_2, N_3; H; L_1, L_2, L_3} \right\|_{[3, \mathbb{R} \times \mathbb{R}]} \lesssim 1. \end{aligned}$$

Here, $X_{N_1, N_2, N_3; H; L_1, L_2, L_3}$ is the multiplier

$$X_{N_1, N_2, N_3; H; L_1, L_2, L_3}(\xi, \tau) := \chi_{|h(\xi)| \sim H} \prod_{j=1}^3 \chi_{|\xi_j| \sim N_j} \chi_{|\lambda_j| \sim L_j}$$

and

$$\tilde{m}(N_1, N_2) := \sup_{|\xi_j| \sim N_j, j=1,2} m(\xi_1, \xi_2)$$

where $m(\xi_1, \xi_2)$ is a multiplier of the bilinear operator $B(\cdot, \cdot)$. This leads us to consider

$$\|X_{N_1, N_2, N_3; H; L_1, L_2, L_3}\|_{[3, \mathbb{R} \times \mathbb{R}]}, \quad (3.1)$$

which vanishes unless

$$N_{\text{med}} \sim N_{\text{max}} \quad (3.2)$$

$$L_{\text{max}} \sim \max(H, L_{\text{med}}) \quad (3.3)$$

Moreover, we have the resonance relation: if $N_{\text{max}} \sim N_{\text{med}} \gtrsim 1$, then

$$H \sim N_{\text{max}}^4 N_{\text{min}} \quad (3.4)$$

Now we state the dyadic block estimate.

Lemma 3.1. *Let $H, N_1, N_2, N_3, L_1, L_2, L_3 > 0$ satisfy (3.2), (3.4), (3.3).*

(a) ((++)Coherence) *If $N_{\text{max}} \sim N_{\text{min}}$ and $L_{\text{max}} \sim H$, then we have*

$$(3.1) \lesssim L_{\text{min}}^{1/2} N_{\text{max}}^{-2} L_{\text{med}}^{1/2}. \quad (3.5)$$

(b) ((+-)Coherence) *If $N_2 \sim N_3 \gg N_1$ and $H \sim L_1 \gtrsim L_2, L_3$, then*

$$(3.1) \lesssim L_{\text{min}}^{1/2} N_{\text{max}}^{-2} \min\left(H, \frac{N_{\text{max}}}{N_{\text{min}}} L_{\text{med}}\right)^{1/2}. \quad (3.6)$$

Similarly for permutations.

(c) *In all other cases, we have*

$$(3.1) \lesssim L_{\text{min}}^{1/2} N_{\text{max}}^{-2} \min(H, L_{\text{med}})^{1/2}. \quad (3.7)$$

Lemma 3.1 is obtained in a similar way to Tao's ([10], Proposition 6.1) in the context of the KdV equation. For the fifth order equation, it is first shown by Chen, Li, Miao and Wu [2]. See [2] for the proof.

Bilinear estimates. Using Lemma 3.1 we show three bilinear estimates to which the trilinear estimate is reduced.

Proposition 3.2. *For Schwartz functions u, v on $\mathbb{R} \times \mathbb{R}$ and $0 < \epsilon \ll 1$, we have*

$$\|uv\|_{L^2(\mathbb{R} \times \mathbb{R})} \lesssim \|u\|_{X_{\tau=\xi^5}^{-3/2, 1/2-\epsilon}} \|v\|_{X_{\tau=\xi^5}^{3/4, 1/2+\epsilon}}, \tag{3.8}$$

$$\|uv\|_{L^2(\mathbb{R} \times \mathbb{R})} \lesssim \|u\|_{X_{\tau=\xi^5}^{-3/4, 1/2-\epsilon}} \|v\|_{X_{\tau=\xi^5}^{0, 1/2+\epsilon}}, \tag{3.9}$$

$$\|uv\|_{L^2(\mathbb{R} \times \mathbb{R})} \lesssim \|u\|_{X_{\tau=\xi^5}^{-1/4, 1/2-\epsilon}} \|v\|_{X_{\tau=\xi^5}^{-1/2, 1/2+\epsilon}}. \tag{3.10}$$

Proof. We prove (3.8) first. Rewriting (3.8) by duality, Plancherel’s theorem and dyadic decomposition and using the translation invariance of the $[k; Z]$ -multiplier (may assume $L_1, L_2, L_3 \gtrsim 1$ and $\max(N_1, N_2, N_3) \gtrsim 1$) and Schur’s test [10, Lemma 3.11], it suffices to show

$$\begin{aligned} & \sum_{N \sim N_{\max} \sim N_{\text{med}}} \sum_{\substack{L_1, L_2, L_3 \geq 1, \\ H \sim L_{\max}}} \frac{\langle N_2 \rangle^{3/2}}{\langle N_1 \rangle^{3/4} L_1^{1/2+\epsilon} L_2^{1/2-\epsilon}} \\ & \times \|X_{N_1, N_2, N_3; L_{\max}; L_1, L_2, L_3}\|_{[3, \mathbb{R} \times \mathbb{R}]} \lesssim 1 \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} & \sum_{N \sim N_{\max} \sim N_{\text{med}}} \sum_{\substack{L_{\max} \sim L_{\text{med}}, \\ H \ll L_{\max}}} \frac{\langle N_2 \rangle^{3/2}}{\langle N_1 \rangle^{3/4} L_1^{1/2+\epsilon} L_2^{1/2-\epsilon}} \\ & \times \|X_{N_1, N_2, N_3; L_{\max}; L_1, L_2, L_3}\|_{[3, \mathbb{R} \times \mathbb{R}]} \lesssim 1 \end{aligned} \tag{3.12}$$

for all $N \gtrsim 1$. Fix N . We prove (3.12) first. From (3.7) it reduces to show

$$\sum_{N \sim N_{\max} \sim N_{\text{med}}} \sum_{L_{\max} \sim L_{\text{med}} \gtrsim N^4 N_{\min}} \frac{\langle N_2 \rangle^{3/2}}{\langle N_1 \rangle^{3/4} L_1^{1/2+\epsilon} L_2^{1/2-\epsilon}} L_{\min}^{1/2} N^{-2} N^2 N_{\min}^{1/2} \lesssim 1$$

Estimating

$$\begin{aligned} & \frac{\langle N_2 \rangle^{3/2}}{\langle N_1 \rangle^{3/4}} \lesssim \frac{N^{3/2}}{\langle N_{\min} \rangle^{3/4}} \\ & L_1^{1/2+\epsilon} L_2^{1/2-\epsilon} \gtrsim L_{\min}^{1/2+\epsilon} L_{\text{med}}^{\epsilon} (N^4 N_{\min})^{1/2-2\epsilon} \end{aligned}$$

and then performing the L summations, we reduce to

$$\sum_{N \sim N_{\max} \sim N_{\text{med}}} \frac{\langle N \rangle^{3/2} N_{\min}^{1/2}}{\langle N_{\min} \rangle^{3/4} (N^4 N_{\min})^{1/2-\epsilon}} \lesssim 1,$$

which is true with about $N^{-1/2}$ to spare.

Now, we show the case (3.11). In this case we have $L_{\max} \sim N_{\max}^4 N_{\min}$. We first show when (3.5) (i.e. $(++)$ -coherence) holds. From (3.5) we have $N_{\max} \sim N_{\text{med}} \sim N_{\min}$ and (3.1) $\lesssim L_{\min}^{1/2} N_{\max}^{-2} L_{\text{med}}^{1/2}$, so we reduce to

$$\sum_{L_{\max} \sim N^5} \frac{N^{3/2}}{N^{3/4} L_1^{1/2-\epsilon} L_2^{1/2+\epsilon}} L_{\min}^{1/2} N^{-2} L_{\text{med}}^{1/2} \lesssim 1.$$

Estimating

$$L_1^{1/2+\epsilon} L_2^{1/2-\epsilon} \geq L_{\min}^{1/2+\epsilon} L_{\text{med}}^{1/2-\epsilon}$$

and then performing the L summations we reduce to

$$\frac{N^{3/2}}{N^{3/4}N^{5\epsilon}}N^{-2} \lesssim 1,$$

which is true.

Now we deal with (+-)coherence case (i.e. when (3.6) holds true). Since we don't have the symmetry on indices, we need to consider the following three cases:

$$\begin{aligned} N \sim N_1 \sim N_2 \gg N_3; \quad H \sim L_3 \gtrsim L_1, L_2 \\ N \sim N_2 \sim N_3 \gg N_1; \quad H \sim L_1 \gtrsim L_2, L_3 \\ N \sim N_1 \sim N_3 \gg N_2; \quad H \sim L_2 \gtrsim L_1, L_3 \end{aligned}$$

In the first case we reduce by (3.6) to

$$\sum_{N_3 \ll N, L_1, L_2 \lesssim N^4 N_3} \frac{N^{3/2}}{N^{3/4}L_1^{1/2+\epsilon}L_2^{1/2-\epsilon}}L_{\min}^{1/2}N^{-2} \min(N^4 N_3, \frac{N}{N_3}L_{\text{med}})^{1/2} \lesssim 1.$$

Performing the N_3 summation we reduce to

$$\sum_{1 \leq L_1, L_2 \lesssim N^5} \frac{N^{3/2}}{N^{3/4}L_1^{1/2+\epsilon}L_2^{1/2-\epsilon}}L_{\min}^{1/2}N^{-2}N^{5/4}L_{\text{med}}^{1/4} \lesssim 1$$

which is easily verified.

To symmetrize the second and third case we replace $L_1^{1/2+\epsilon}$ by $L_1^{1/2-\epsilon}$. It suffices to show the second case. Using $\min(H, \frac{N}{N_{\min}}L_{\text{med}}) \leq H \sim N^4 N_1$ we reduce to

$$\sum_{N_1 \leq N} \sum_{L_2, L_3 \leq N^4 N_1} \frac{N^{3/2}N_1^{1/2}}{\langle N_1 \rangle^{3/4}(N^4 N_1)^{1/2+\epsilon}L_2^{1/2-\epsilon}} \lesssim 1$$

We may assume $N_1 \geq N^{-4}$ since the inner sum vanishes otherwise. Performing the L summations we reduce to

$$\sum_{N^{-4} \leq N_1 \leq N} N^{3/2-2+4\epsilon} \frac{N_1^\epsilon}{\langle N_1 \rangle^{3/4}}(N^4 N_1)^\epsilon \lesssim 1$$

which is true with about $N^{-1/2}$ to spare. Finally, we show the cases (3.7) holds. It suffices to show

$$\sum_{N_{\max} \sim N_{\text{med}} \sim N} \sum_{L_{\max} \sim N^4 N_{\min}} \frac{N^{3/2}}{\langle N_1 \rangle^{3/4}L_1^{1/2+\epsilon}L_2^{1/2-\epsilon}}L_{\min}^{1/2}N^{-2}L_{\text{med}}^{1/2} \lesssim 1$$

Performing the L summations, we reduce to

$$\sum_{N_{\max} \sim N_{\text{med}} \sim N} \frac{N^{-1/2}}{\langle N_1 \rangle^{3/4}}(N^4 N_{\min})^\epsilon \lesssim 1$$

which is easily verified with about $N^{-1/2}$ to spare. This completes the proof for (3.8). The proof of (3.9) and (3.10) are very similar to the preceding one. In general, the same computation shows

$$\|uv\|_{L^2(\mathbb{R} \times \mathbb{R})} \lesssim \|u\|_{X_{\tau=\xi^5}^{-\alpha, 1/2-\epsilon}} \|v\|_{X_{\tau=\xi^5}^{\beta, 1/2+\epsilon}}.$$

for $\alpha < 2$ and $\alpha - \beta \leq 3/4$. We omit the detail here. \square

Proof of the trilinear estimate. To reduce the trilinear estimate we use the following lemma.

Lemma 3.3 (Tao [10, Lemma 3.7] Composition and TT^*). *If $k_1, k_2 \geq 1$, and m_1, m_2 are functions on \mathbb{R}^{k_1} and \mathbb{R}^{k_2} respectively, then*

$$\begin{aligned} & \|m_1(\xi_1, \dots, \xi_{k_1})m_2(\xi_{k_1+1}, \dots, \xi_{k_1+k_2})\|_{[k_1+k_2;\mathbb{R}]} \\ & \leq \|m_1(\xi_1, \dots, \xi_{k_1})\|_{[k_1+1;\mathbb{R}]} \|m_2(\xi_1, \dots, \xi_{k_2})\|_{[k_2+1;\mathbb{R}]} . \end{aligned} \tag{3.13}$$

As a special case we have the TT^* identity

$$\|m(\xi_1, \dots, \xi_k) \overline{m(-\xi_{k+1}, \dots, -\xi_{2k})}\|_{[2k;\mathbb{R}]} = \|m(\xi_1, \dots, \xi_k)\|_{[k+1;\mathbb{R}]}^2 \tag{3.14}$$

for all functions $m : \mathbb{R}^k \rightarrow \mathbb{R}$.

For simplicity we prove the most interesting case $s = 3/4$. For the first term it suffices to show that

$$\left\| \frac{\langle \xi_1 + \xi_2 + \xi_3 \rangle^3 \langle \xi_4 \rangle^{3/4}}{\langle \tau_4 - \xi_4^5 \rangle^{1-b} \prod_{j=1}^3 \langle \xi_j \rangle^s \langle \tau_j - \xi_j^5 \rangle^b} \right\|_{[4, \mathbb{R} \times \mathbb{R}]} \lesssim 1.$$

Estimating $|\xi_1 + \xi_2 + \xi_3|$ by $\langle \xi_4 \rangle$, and

$$\langle \xi_4 \rangle^{3/4+3} \lesssim \langle \xi_4 \rangle^{3/2} \sum_{j=1}^3 \langle \xi_j \rangle^{3/4+3/2}.$$

By symmetry we reduce to

$$\left\| \frac{\langle \xi_1 \rangle^{-3/4} \langle \xi_3 \rangle^{-3/4} \langle \xi_2 \rangle^{3/2} \langle \xi_4 \rangle^{3/2}}{\langle \tau_4 - \xi_4^5 \rangle^{1-b} \prod_{j=1}^3 \langle \tau_j - \xi_j^5 \rangle^b} \right\|_{[4, \mathbb{R} \times \mathbb{R}]} \lesssim 1.$$

We may replace $\langle \tau_2 - \xi_2^5 \rangle^b$ by $\langle \tau_2 - \xi_2^5 \rangle^{1-b}$. By TT^* identity (3.14), the estimate is reduced to the bilinear estimate (3.8).

The proof of the second term (2.2) is very similar to the first one but we use the composition rule (3.13) instead of the TT^* identity. We estimate

$$\xi_1^3 \leq \xi_1^{9/4} \left(\langle \xi_2 \rangle^{3/4} + \langle \xi_3 \rangle^{3/4} + \langle \xi_4 \rangle^{3/4} \right) \langle \xi_4 \rangle^{3/4}.$$

The third term is the same as above and so by symmetry we reduce to

$$\left\| \frac{\langle \xi_1 \rangle^{3/2} \langle \xi_3 \rangle^{-3/4} \langle \xi_2 \rangle^0 \langle \xi_4 \rangle^{3/4}}{\langle \tau_4 - \xi_4^5 \rangle^{1-b} \prod_{j=1}^3 \langle \tau_j - \xi_j^5 \rangle^b} \right\|_{[4, \mathbb{R} \times \mathbb{R}]} \lesssim 1.$$

This is verified by (3.8) and (3.9), as well as the composition rule (3.13).

The fourth term in (2.2) is proved in the same way. Estimating

$$\langle \xi_4 \rangle^{3/4} \leq \langle \xi_4 \rangle^{1/2} (\langle \xi_1 \rangle^{1/4} + \langle \xi_2 \rangle^{1/4} + \langle \xi_3 \rangle^{1/4}),$$

and by symmetry we reduce to

$$\left\| \frac{\langle \xi_1 \rangle^{1/2} \langle \xi_2 \rangle^{1/4} \langle \xi_3 \rangle^{1/4} \langle \xi_4 \rangle^{1/2}}{\langle \tau_4 - \xi_4^5 \rangle^{1-b} \prod_{j=1}^3 \langle \tau_j - \xi_j^5 \rangle^b} \right\|_{[4, \mathbb{R} \times \mathbb{R}]} \lesssim 1.$$

This is verified by (3.10) and TT^* identity (3.14) after minorizing one of b by $1 - b$. Finally, the third term in (2.2) automatically follows since it is a linear combination of other three. This concludes the proof of Proposition 2.3.

Remark 3.4. The trilinear estimate (2.2) fails for $s < \frac{3}{4}$. The counter example introduced by Kenig, Ponce and Vega [7] in the context of the modified KdV equation extends to here. In the frequency space, set

$$A = \{(\tau, \xi) \in \mathbb{R}^2 \mid N \leq \xi \leq N + N^{-3/2}, |\tau - \xi^5| \leq 1\},$$

and

$$-A = \{(\tau, \xi) \in \mathbb{R}^2 \mid -(\tau, \xi) \in A\}.$$

Defining $\tilde{f}(\tau, \xi) = \chi_A + \chi_{-A}$, we obtain

$$|\tilde{f} * \tilde{f} * \tilde{f}(\tau, \xi)| \gtrsim N^{-3} \chi_R(\tau, \xi),$$

where R is a rectangle located at (N, N^5) of dimension $N^{-4} \times N^{5/2}$ with its longest side pointing $(1, 5N^4)$ like A . Thus,

$$\|\partial_x^3(f \cdot f \cdot f)\|_{X^{s, b-1}} \gtrsim N^{s-3/4}$$

and

$$\|f\|_{X^{s, b}} \lesssim N^{s-3/4},$$

then (2.2) implies $s \geq 3/4$. This example holds good for other nonlinear terms in (2.2).

Remark 3.5. In our general equation (1.2) we omitted the term $u^4 \partial_x u$ from (1.1). Since the term $u^4 \partial_x u$ is a lower order term, it is easier to handle than other third order terms. Once we have the 5-linear estimate

$$\|\partial_x(u_1 u_2 u_3 u_4 u_5)\|_{X^{s, b-1}} \lesssim \prod_{j=1}^5 \|u_j\|_{X^{s, b}}, \quad (3.15)$$

we can insert it into the iteration. The proof of (3.15) is similar to the preceding one. Using Lemma 3.3 we reduce to two trilinear estimates and each trilinear estimate is again reduced to two bilinear estimates. The resulting bilinear estimates are supposedly easier than those in Proposition 3.2 since there are fewer derivatives and more u 's. In fact, it is true for s lower than $3/4$.

4. ILL-POSEDNESS

In this section we give the proof of Theorem 1.2. For simplicity, we pretend the nonlinear term is

$$F(u) = \partial_x^3(u^3).$$

The general case $F(u) = c_1 \partial_x^3(u^3) + c_2 u^2 \partial_x^3 + c_3 u \partial_x u \partial_x^2$ (c_j 's are real numbers) follows in the same manner. Our method is to approximate the fifth mKdV solution by the cubic NLS solution. This is originally introduced by Christ, Colliander and Tao [3] for the mKdV equation. This method extends to the fifth order equation without substantial change.

Having two solutions to the cubic NLS breaking the uniform continuity of the flow map for $s < 0$, we find approximate solutions to the fifth mKdV exhibiting the same property. First, we state the ill-posedness for the cubic NLS in [3].

Theorem 4.1. *Let $s < 0$. The solution map of the initial value problem of the cubic NLS (1.6) fails to be uniformly continuous. More precisely, for $0 < \delta \ll \epsilon \ll 1$ and*

$T > 0$ arbitrary, there are two solutions u_1, u_2 to (1.6) satisfying (1.3), (1.4) and (1.5). Moreover, For any fixed $K \geq 1$, we can find such solutions to satisfy

$$\sup_{0 \leq t < \infty} \|u_j\|_{H_x^K} \lesssim \epsilon \quad (4.1)$$

for $j = 1, 2$.

Remark 4.2. Theorem 4.1 is stated for the defocusing cubic NLS. The method in [3] exhibiting the phase decoherence holds good for the focusing case, too. But previously another method for the focusing case was presented by Kenig, Ponce and Vega [8]. They used the Galilean invariance on the soliton solutions. In our focusing case (for instance, $F(u) = -\partial_x^3(u^3)$) one could employ their counterexample to approximate.

Now we start to find the approximate solution to the fifth order mKdV equation using the NLS solutions. Let $u(s, y)$ solve the cubic NLS equation (1.6). We also assume that

$$\sup_{0 \leq t < \infty} \|u(t)\|_{H_x^k} \lesssim \epsilon$$

for a large k . Using the change of variable

$$(s, y) := \left(t, \frac{x}{(10N^3)^{1/2}} + \sqrt{\frac{5}{2}} N^{5/2} t \right),$$

we define the approximate solution

$$U_{ap}(t, x) := \frac{2}{\sqrt{3N^3}} \operatorname{Re} e^{iNx} e^{iN^5 t} u(s, y), \quad (4.2)$$

where $N \gg 1$. We want to show that U_{ap} is an approximate solution to the fifth mKdV equation. A direct computation shows that

$$\begin{aligned} (\partial_t - \partial_x^5) U_{ap}(t, x) &= \frac{2}{\sqrt{3N^3}} \operatorname{Re} \left\{ e^{iNx} e^{iN^5 t} \left(\partial_s u + i \partial_y^2 u \right. \right. \\ &\quad \left. \left. + \frac{1}{\sqrt{10N^{5/2}}} \partial_y^3 u - \frac{i}{20N^5} \partial_y^4 u - \frac{1}{(10N^3)^{5/2}} \partial_y^5 u \right) \right\} \end{aligned}$$

and that

$$\begin{aligned} &\partial_x^3 (U_{ap}^3) \\ &= \left(\frac{2}{\sqrt{3N^3}} \right)^3 \frac{3}{4} \partial_x^3 \left\{ \operatorname{Re} e^{iNx} e^{iN^5 t} |u|^2 u + \frac{1}{3} \operatorname{Re} e^{iNx} e^{iN^5 t} u^3 \right\} \\ &= \left(\frac{2}{\sqrt{3N^3}} \right)^3 \frac{3}{4} \left\{ \operatorname{Re} (iN)^3 e^{iNx} e^{iN^5 t} |u|^2 u + \operatorname{Re} \frac{3(iN)^2}{\sqrt{10N^{3/2}}} e^{iNx} e^{iN^5 t} \partial_y (|u|^2 u) \right. \\ &\quad + \operatorname{Re} \frac{3iN}{10N^3} e^{iNx} e^{iN^5 t} \partial_y^2 (|u|^2 u) + \operatorname{Re} \frac{1}{(10N^3)^{3/2}} e^{iNx} e^{iN^5 t} \partial_y^3 (|u|^2 u) \\ &\quad + \operatorname{Re} \frac{(3iN)^3}{3} e^{3iNx} e^{3iN^5 t} u^3 + \operatorname{Re} \frac{(3iN)^2}{\sqrt{10N^{3/2}}} e^{3iNx} e^{3iN^5 t} \partial_y (u^3) \\ &\quad \left. + \operatorname{Re} \frac{3iN}{10N^3} e^{3iNx} e^{3iN^5 t} \partial_y^2 (u^3) + \operatorname{Re} \frac{1}{3(10N^3)^{3/2}} e^{3iNx} e^{3iN^5 t} \partial_y^3 (u^3) \right\}. \end{aligned}$$

Since $u(s, y)$ is a solution of (1.6), three terms of the preceding equations canceled and it results in

$$(\partial_t - \partial_x^5) U_{ap}(t, x) + \partial_x^3 (U_{ap}^3) = E$$

where the error term E is a linear combination of the real and imaginary parts of the following:

$$\begin{aligned} E_1 &:= N^{-4}e^{iNx}e^{iN^5t}\partial_y(|u|^2u), & E_2 &:= N^{-11/2}e^{iNx}e^{iN^5t}\partial_y^2(|u|^2u), \\ E_3 &:= N^{-18/2}e^{iNx}e^{iN^5t}\partial_y^3(|u|^2u), & E_4 &:= N^{-4}e^{3iNx}e^{3iN^5t}\partial_y(u^3), \\ E_5 &:= N^{-11/2}e^{3iNx}e^{3iN^5t}\partial_y^2(u^3), & E_6 &:= N^{-18/2}e^{iNx}e^{iN^5t}\partial_y^3(u^3), \\ E_7 &:= N^{-3/2}e^{3iNx}e^{3iN^5t}u^3. \end{aligned}$$

Next, we find a bound of the error.

Lemma 4.3. *For each $j = 1, \dots, 7$, let e_j be the solution to the initial problem*

$$(\partial_t - \partial_x^5)e_j = E_j; \quad e_j(0) = 0.$$

Let $\eta(t)$ be a smooth time cut-off function taking 1 on $[0, 1]$ and compactly supported. Then

$$\|\eta(t)e_j\|_{X^{3/4,b}} \lesssim \epsilon N^{-5/2+\delta}$$

for arbitrarily small $\delta > 0$.

For the proof we use the estimate of high-frequency modulations of smooth functions.

Lemma 4.4 ([3, Lemma 2.1]). *Let $-1/2 < s, \sigma \in \mathbb{R}^+$ and $u \in H^\sigma(\mathbb{R})$. For any $M > 1, \tau \in \mathbb{R}^+, x_0 \in \mathbb{R}$, and $A > 0$ let*

$$v(x) = Ae^{iMx}u\left(\frac{x-x_0}{\tau}\right).$$

- (i) *Suppose $s \geq 0$. Then $\|v\|_{H^1} \lesssim_s |A|\tau^{1/2}M^s\|u\|_{H^s}$ for all u, A, x_0 and $M \cdot \tau \geq 1$.*
- (ii) *Suppose that $s < 0$ and that $\sigma \geq |s|$. Then $\|v\|_{H^s} \lesssim_{s,\sigma} |A|\tau^{1/2}M^s\|u\|_{H^\sigma}$ for all u, A, x_0 and $M^{1+(s/\sigma)} \cdot \tau \geq 1$.*

Proof of Lemma 4.3. Using (2.1) and Plancherel theorem we obtain

$$\begin{aligned} \|\eta(t)e_j\|_{X^{3/4,b}} &\lesssim \|\eta(t)E_j\|_{X^{3/4,b-1}} \\ &= \|\langle \tau - \xi^5 \rangle^{b-1} \langle \xi \rangle^{3/4} \widetilde{\eta(t)E_j}\|_{L_{\tau,\xi}^2} \\ &\leq \|\langle \xi \rangle^{3/4} \widetilde{\eta(t)E_j}\|_{L_{\tau,\xi}^2} \quad (\because b-1 < 0) \\ &= \|\eta(t)\langle \xi \rangle^{3/4} \widehat{E_j}(t, \xi)\|_{L_{t,\xi}^2} \\ &\leq \|\langle \xi \rangle^{3/4} \widehat{E_j}(t, \xi)\|_{L_t^\infty L_x^2([0,1] \times \mathbb{R})} \end{aligned}$$

Thus, we reduce to show

$$\sup_{0 \leq t \leq 1} \|E_j\|_{H_x^{3/4}} \lesssim \epsilon N^{-5/2+\delta}.$$

E_1, \dots, E_6 have enough negative powers of N . The above bound for these terms is obtained by (4.1), Lemma 4.4 and the fact that H^k is closed under multiplication for $k \geq 1$. For the last term E_7 since there is not enough of a negative power on N , we need to use the fact that the modulation $e^{3iNx}e^{3iN^5t}$ is away from the curve $\tau = \xi^5$. A direct computation leads that

$$\widetilde{\eta(t)E_7}(\tau, \xi) = N^{-3/2} \widetilde{\eta} u^3 \left(\tau - a, \sqrt{10}N^{3/2}(\xi - 3N) \right) \sqrt{10}N^{3/2}$$

where $a = 3N^5 - 3\sqrt{\frac{5}{2}}N^5 + \sqrt{\frac{5}{2}}N^4\xi$.

Let $P_{\lambda,\mu}$ be the Littlewood-Paley projection with dyadic numbers λ, μ . (4.1) and the fact that $\eta(t)$ is compactly supported yield

$$\|P_{\lambda,\mu}\widetilde{\eta}u^3(\tau, \xi)\|_{L^2_{\tau,\xi}} \lesssim \frac{\epsilon}{\langle \lambda \rangle^K \langle \mu \rangle^K}$$

and so

$$\|P_{\lambda,\mu}\widetilde{\eta}u^3(\tau - a, N^{3/2}(\xi - 3N))\|_{L^2_{\tau,\xi}} \lesssim N^{-3/4} \frac{\epsilon}{\langle \lambda - a \rangle^K \langle \mu - 3N \rangle^K}.$$

Rewriting $\|\eta(t)E_7\|_{X^{3/4,b-1}}$ by dyadic decompositions,

$$\begin{aligned} \|\eta(t)E_7\|_{X^{3/4,b-1}}^2 &\lesssim \sum_{\substack{\lambda,\mu \geq 1 \\ \text{dyadic}}} \langle \lambda - \mu^5 \rangle^{2(b-1)} \langle \mu \rangle^{3/2} N^{-3} \\ &\quad \times \left\| P_{\lambda,\mu}\widetilde{\eta}u^3\left(\tau - a, \sqrt{10}N^{3/2}(\xi - 3N)\right) \sqrt{10}N^{3/2} \right\|_{L^2_{\tau,\xi}}^2 \\ &\lesssim \sum_{\substack{\lambda,\mu \geq 1 \\ \text{dyadic}}} \langle \lambda - \mu^5 \rangle^{2(b-1)} \langle \mu \rangle^{3/2} N^{-3/2} \frac{\epsilon^2}{\langle \lambda - a \rangle^{2K} \langle \mu - 3N \rangle^{2K}} \\ &\lesssim \epsilon^2 N^{10(b-1)} \end{aligned}$$

by choosing K large enough. We used the fact that $e^{3iNx}e^{3iN^5t}$ is away from the curve in the frequency space at the last inequality. Therefore, choosing $b > \frac{1}{2}$ sufficiently close to $\frac{1}{2}$ we conclude

$$\|\eta(t)E_7\|_{X^{3/4,b-1}}^2 \lesssim \epsilon N^{-5/2+\delta}.$$

□

Finally, we state the following perturbation result from the local well-posedness.

Lemma 4.5. *Let u be a Schwartz solution to the fifth order modified KdV equation (1.2) and v be a Schwartz solution to the approximate fifth mKdV equation*

$$\partial_t v - \partial_x^5 v + \partial_x^3(v^3) = E$$

for some error function E . Let e be the solution to the inhomogeneous problem

$$\partial_t e - \partial_x^5 e = E, \quad e(0) = 0.$$

Suppose that

$$\|u(0)\|_{H_x^{3/4}}, \|v(0)\|_{H_x^{3/4}} \lesssim \epsilon; \quad \|\eta(t)e\|_{X^{3/4,b}} \lesssim \epsilon$$

Then we have

$$\|\eta(t)(u - v)\|_{X^{3/4,b}} \lesssim \|u(0) - v(0)\|_{H^{3/4}} + \|\eta(t)e\|_{X^{3/4,b}}.$$

In particular, we have

$$\sup_{0 \leq t \leq 1} \|u(t) - v(t)\|_{H^{3/4}} \lesssim \|u(0) - v(0)\|_{H^{3/4}} + \|\eta(t)e\|_{X^{3/4,b}}.$$

Proof. The proof is very similar to that of [3, Lemma 5.1]. Here, we give only a sketch. Writing the integral equation for v with a time cut-off function $\eta(t)$,

$$\eta(t)v(t) = \eta(t)e^{t\partial_x^5}v(0) - \eta(t)e(t) + \eta(t) \int_0^t e^{(t-t')\partial_x^5} \partial_x^3(v^3)(t') dt'.$$

we use (2.1), (2.2) and a continuity argument (assuming ϵ is sufficiently small) to obtain

$$\|\eta(t)v\|_{X^{3/4,b}} \lesssim \epsilon.$$

We repeat the same argument on the difference of the two $w = u - v$ to get the desired result. \square

Proof of Theorem 1.2. Let $0 < \delta \ll \epsilon \ll 1$ and $T > 0$ be given. From Theorem 4.1 we can find two global solutions u_1, u_2 satisfying

$$\|u_j(0)\|_{H_x^s} \lesssim \epsilon \tag{4.3}$$

$$\|u_1(0) - u_2(0)\|_{H_x^s} \lesssim \delta \tag{4.4}$$

$$\sup_{0 \leq t \leq T} \|u_1(t) - u_2(t)\|_{H_x^s} \gtrsim \epsilon \tag{4.5}$$

$$\sup_{0 \leq t \leq \infty} \|u_j(t)\|_{H_x^k} \lesssim \epsilon \tag{4.6}$$

for $s < 0$ and $k \geq 6$ to be chosen later. Define $U_{ap,1}$ and $U_{ap,2}$ as in (4.2), and let U_1, U_2 be smooth global solutions with initial data $U_{ap,1}, U_{ap,2}$, respectively. Now we rescale these solutions to make them satisfy (1.3), (1.4), (1.5). Set

$$U_j^\lambda(t, x) := \lambda U_j(\lambda^5 t, \lambda x)$$

and similarly,

$$U_{ap,j}^\lambda(t, x) := \lambda U_{ap,j}(\lambda^5 t, \lambda x),$$

for $j = 1, 2$. Then

$$U_j^\lambda(0, x) = \lambda \frac{2}{\sqrt{3N^3}} \operatorname{Re} e^{iN\lambda x} u(0, \lambda x / (10N^3)^{1/2}).$$

From Lemma 4.3 and Lemma 4.5 we have

$$\sup_{0 \leq t \leq 1} \|U_1(t) - U_2(t)\|_{H_x^{3/4}} \lesssim \epsilon N^{-5/2+\delta}.$$

An induction argument on time interval up to $\log N$ yields

$$\sup_{0 \leq t \lesssim \eta \log N} \|U_1(t) - U_2(t)\|_{H_x^{3/4}} \lesssim \epsilon N^{-5/2+\eta} \tag{4.7}$$

for any $\eta > \delta > 0$. Applying Lemma 4.4 when $s \geq 0$ we obtain

$$\|U_j^\lambda(0)\|_{H_x^s} \lesssim \lambda^{s+1/2} N^{s-3/4} \|u_j(0)\|_{H_x^s},$$

while for $s < 0$, we use Lemma 4.4 (ii) for sufficiently large k to obtain

$$\|U_j^\lambda(0)\|_{H_x^s} \lesssim \lambda^{s+1/2} N^{s-3/4} \|u_j(0)\|_{H_x^k}.$$

Setting $\lambda := N^{\frac{3/4-s}{17/2+s}}$, and from (4.3), (4.6) we have (1.3) for $U_j^\lambda(0)$. Similarly, we also get (1.4) for $U_1^\lambda(0) - U_2^\lambda(0)$ from (4.4).

Next, we show (1.5). From (4.5) one can find $0 < t_0$ such that

$$\|u_1(t_0) - u_2(t_0)\|_{L_x^2} \gtrsim \epsilon.$$

Using Lemma 4.4 we obtain

$$\|U_{ap,1}(t_0/\lambda^5) - U_{ap,2}(t_0/\lambda^5)\|_{H_x^s} \gtrsim \lambda^{1/2+s} N^{s-3/4} \epsilon \sim \epsilon.$$

On the other hand, using the hypothesis $s > -\frac{7}{24}$ and (4.7)

$$\|U_{ap,j}^\lambda(t) - U_j^\lambda(t)\|_{H^s} \lesssim \lambda^{\max(0,s)+1/2} \epsilon N^{-5/2+\eta} \lesssim \epsilon$$

for $0 < t \lesssim_{\eta} \log N/\lambda^5$ and sufficiently small $\eta > 0$. A triangle inequality shows

$$\|U_1^\lambda(t_0/\lambda^5) - U_2^\lambda(t_0/\lambda^5)\|_{H_x^s} \gtrsim \epsilon$$

for $t_0/\lambda^5 \ll \log N/\lambda^5$. Choosing λ (and hence N) large enough that $t_0/\lambda^5 < T$, we get (1.5). This completes the proof. \square

Acknowledgements. The author would like to appreciate his advisor Terence Tao for many helpful conversations and encouragement.

REFERENCES

- [1] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV-equation*, Geom. Funct. Anal. **3** (1993), no. 3, 209–262.
- [2] W. Chen, J. Li, C. Miao, J. Wu, *Low regularity solution of a 5th-order KdV equation*, preprint, math. AP/0710.2704.
- [3] M. Christ, J. Colliander, and T. Tao, *Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations*, Amer. J. Math. **125** (2003), no. 6, 1235–1293.
- [4] M. Christ, J. Colliander, and T. Tao, *Ill-posedness for nonlinear Schrödinger and wave equations*, to appear Annales IHP, math.AP/0311048.
- [5] C. E. Kenig, G. Ponce, and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math. **46** (1993), 527–620.
- [6] C. E. Kenig, G. Ponce, and L. Vega, *On the hierarchy of the generalized KdV equations*, Singular limits of dispersive waves (Lyon, 1991), 347–356, NATO Adv. Sci. Inst. Ser. B Phys., 320, Plenum, New York, 1994.
- [7] C. E. Kenig, G. Ponce, and L. Vega, *A bilinear estimate with applications to the KdV equation*, J. Amer. Math. Soc. **9** (1996), no. 2, 573–603.
- [8] C. E. Kenig, G. Ponce, and L. Vega, *On the ill-posedness of some canonical dispersive equations*, Duke Math. J. **106** (2001), no. 3, 617–633.
- [9] S. Kwon, *On the fifth order KdV equation: local well-posedness and lack of uniform continuity of the solution map*, preprint, math. AP/0708.4010.
- [10] T. Tao, *Multilinear weighted convolution of L^2 functions, and applications to nonlinear dispersive equations*, Amer. J. Math. **123** (2001), no. 5, 839–908.
- [11] T. Tao, *Nonlinear dispersive equations, local and global analysis*, CBMS Regional Conference Series in Mathematics, American Mathematical Society (2006).

SOONSIK KWON

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CA 90095-1555, USA

E-mail address: rhex2@math.ucla.edu