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ANALYSIS OF TWO DYNAMIC FRICTIONLESS CONTACT PROBLEMS FOR ELASTIC-VISCO-PLASTIC MATERIALS

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Abstract. We consider two mathematical models which describe the contact between an elastic-visco-plastic body and an obstacle, the so-called foundation. In both models the contact is frictionless and the process is assumed to be dynamic. In the first model the contact is described with a normal compliance condition and, in the second one, is described with a normal damped response condition. We derive a variational formulation of the models which is in the form of a system coupling an integro-differential equation with a second order variational equation for the displacement and the stress fields. Then we prove the unique weak solvability of the models. The proofs are based on arguments on nonlinear evolution equations with monotone operators and fixed point. Finally, we study the dependence of the solution with respect to a perturbation of the contact conditions and prove a convergence result.

1. Introduction

Phenomena of contact involving deformable bodies abound in industry and everyday life. Contact of braking pads with wheels, tires with roads, pistons with skirts are just three simple examples. Common industrial processes such as metal forming, metal extrusion, involve contact evolutions. Owing to their inherent complexity, contact phenomena are modelled by nonlinear evolutionary problems.

The aim of this paper is to study two dynamic contact problems for elastic-visco-plastic materials with a constitutive law of the form

$$\boldsymbol{\sigma}(t) = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}(s) - \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s)), \varepsilon(\boldsymbol{u}(s)))\, ds, \tag{1.1}$$

where $\boldsymbol{u}$ denotes the displacement field and $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ represent the stress and the linearized strain tensor, respectively. Here $\mathscr{A}$ and $\mathscr{E}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, and $\mathscr{G}$ is a nonlinear constitutive function which describes the visco-plastic behaviour of the material. In (1.1) and everywhere in this paper the dot above a variable represents derivative with respect to the time variable t.

Examples of constitutive laws of the form (1.1) can be constructed by using rheological arguments, see e.g. [9]. It follows from (1.1) that, at each time moment

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t, the stress tensor $\boldsymbol{\sigma}(t)$ is split into two parts: $\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}^V(t) + \boldsymbol{\sigma}^R(t)$, where $\boldsymbol{\sigma}^V(t) = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))$ represents the purely viscous part of the stress whereas $\boldsymbol{\sigma}^R(t)$ satisfies a rate-type elastic-visco-plastic relation,

$$\boldsymbol{\sigma}^R(t) = \mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}^R(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds. \tag{1.2}$$

When $\mathscr{G} = \mathbf{0}$ the constitutive law (1.1) reduces to the Kelvin-Voigt viscoelastic constitutive relation,

$$\boldsymbol{\sigma} = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}). \tag{1.3}$$

Examples and mechanical interpretation of elastic-visco-plastic materials of the form (1.2) can be found in [6, 10]. Quasistatic contact problems for materials of the form (1.2) or (1.3) are the topic of numerous papers, e.g. [1, 2, 8, 20], and the comprehensive references [9, 22]. Dynamic contact problems with Kelvin-Voigt materials of the form (1.3) are studied in in [11, 12, 15, 16, 18] and in the monograph [7].

The two problems we consider in this paper are frictionless. In the first one we assume that the normal stress on the contact surface depends only on the normal displacement and therefore we model the contact with normal compliance. The normal compliance contact condition was first considered in [19] in the study of dynamic problems with linearly elastic and viscoelastic materials and then it was used in various papers, see e.g. [3, 4, 13, 14, 20] and the references therein. This condition allows the interpenetration of the body's surface into the obstacle and it was justified by considering the interpenetration and deformation of surface asperities. In the second problem we assume that the normal stress on the contact surface depends only on the normal velocity and therefore we model the contact with normal damped response. Such condition is appropriate when the contact surfaces are lubricated; it was used in a number of papers, see, e.g. [9, 17, 21, 22] and the references therein.

The paper is structured as follows. In Section 2 we introduce some notation and preliminaries. In Section 3 we describe the two mathematical models for the frictionless contact precess. In Section 4 we list the assumption on the data and derive the variational formulation of the problems, which is in the form of a nonlinear integro-differential system for the displacement and the stress fields. Then, we state our main existence and uniqueness results, Theorems 4.1 and 4.2. The proof of the theorems is provided in Sections 5 and are based on arguments of abstract evolution equations with monotone operators and fixed point. In Section 6 we study the dependence of the solution with respect to a perturbation of the contact conditions and prove a convergence result.

2. Notation and preliminaries

In this section we present the notation we shall use and some preliminary material. For further details, we refer the reader to [9, 22].

We denote by $\mathscr{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$, while " $\cdot$ " and $\|\cdot\|$ will represent the inner product and the Euclidean norm on $\mathscr{S}^d$ and $\mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let $\boldsymbol{\nu}$ denote the unit outer normal on Γ. Everywhere in the sequel the index i and j run from 1 to d, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent spatial variable.

We use the standard notation for Lebesgue and Sobolev spaces associated to Ω and Γ and introduce the spaces

$$\begin{aligned}
\mathcal{H} &= \{\boldsymbol{\sigma} = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},\\
H_1 &= \{\boldsymbol{u} = (u_i) : \boldsymbol{\varepsilon}(\boldsymbol{u}) \in \mathcal{H}\},\\
\mathcal{H}_1 &= \{\boldsymbol{\sigma} \in \mathcal{H} : \operatorname{Div} \boldsymbol{\sigma} \in L^2(\Omega)^d\}.
\end{aligned}$$

Here $\boldsymbol{\varepsilon}$ and Div are the deformation and the divergence operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \operatorname{Div} \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The spaces $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$\begin{aligned}
(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} &= \int_\Omega \sigma_{ij}\tau_{ij}dx,\\
(\boldsymbol{u}, \boldsymbol{v})_{H_1} &= (\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega)^d} + (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}},\\
(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} &= (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \operatorname{Div} \boldsymbol{\tau})_{L^2(\Omega)^d}.
\end{aligned}$$

In general, we denote by $\|\cdot\|_X$ the norm on a Banach space X and note that this holds, in particular, for the associated norms on the spaces $\mathcal{H}$, H_1 and $\mathcal{H}_1$.

For every element $\boldsymbol{v} \in H_1$ we also use the notation $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on Γ and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and the tangential components of $\boldsymbol{v}$ on Γ given by

$$v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}, \quad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}.$$

We also denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and the tangential traces of a function $\boldsymbol{\sigma} \in \mathcal{H}_1$, and we recall that when $\boldsymbol{\sigma}$ is a regular function then

$$\sigma_\nu = (\boldsymbol{\sigma\nu}) \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu},$$

and the following Green's formula holds:

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \boldsymbol{v})_{L^2(\Omega)^d} = \int_\Gamma \boldsymbol{\sigma\nu} \cdot \boldsymbol{v}\, da \quad \forall \boldsymbol{v} \in H_1. \tag{2.1}$$

Let Γ_1 be a measurable part of Γ such that $meas\, \Gamma_1 > 0$ and let V be the closed subspace of H_1 given by

$$V = \{\ \boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \ \Gamma_1\ \}.$$

Then, the following Korn's inequality holds:

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} \geq c_K \|\boldsymbol{v}\|_{H_1} \quad \forall\, \boldsymbol{v} \in V, \tag{2.2}$$

where $c_K > 0$ is a constant depending only on Ω and Γ_1. A proof of Korn's inequality can be found in, for instance, [7, p. 16]. Over the space V we consider the inner product given by

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}$$

and let $\|\cdot\|_V$ be the associated norm. It follows from Korn's inequality (2.2) that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent norms on V. Therefore $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a positive constant c_0 which depends on Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \leq c_0\|\boldsymbol{v}\|_V \quad \forall \boldsymbol{v} \in V. \tag{2.3}$$

Let $T > 0$. For every real Banach space X we use the notation $C([0,T];X)$ and $C^1([0,T];X)$ for the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively; $C([0,T];X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X$$

while $C^1([0,T];X)$ is a real Banach space with the norm

$$\|x\|_{C^1([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X + \max_{t\in[0,T]} \|\dot{x}(t)\|_X.$$

Finally, for $k \in \mathbb{N}$ and $p \in [1,\infty]$, we use the standard notation for the Lebesgue spaces $L^p(0,T;X)$ and for the Sobolev spaces $W^{k,p}(0,T;X)$.

We complete this section with the following abstract result which may be found in [5, p. 140] and which will be used in Section 5 of this paper.

Theorem 2.1. *Let $V \subset H \subset V'$ be a Gelfand triple. Assume that $A : V \to V'$ is a hemicontinuous and monotone operator which satisfies*

$$\langle Av, v\rangle_{V'\times V} \geq \omega\|v\|_V^2 + \alpha \quad \forall v \in V, \tag{2.4}$$

$$\|Av\|_{V'} \leq C\left(\|v\|_V + 1\right) \quad \forall v \in V, \tag{2.5}$$

for some constants $\omega > 0$, $C > 0$ and $\alpha \in \mathbb{R}$. Then, given $u_0 \in H$ and $f \in L^2(0,T;V')$, there exists a unique function u which satisfies

$$\begin{gathered} u \in L^2(0,T;V) \cap C([0,T];H),\ \dot{u} \in L^2(0,T;V'), \\ \dot{u}(t) + Au(t) = f(t) \quad a.e.\ t \in (0,T), \\ u(0) = u_0. \end{gathered}$$

3. Statement of the Problems

In this section we present the mathematical models which describe the frictionless contact process between an elastic-visco-plastic body and the foundation.

The physical setting is as follows : an elastic-visco-plastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ with a regular boundary Γ that is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that $meas\,\Gamma_1 > 0$. Let $T > 0$ and let $[0,T]$ denote the time interval of interest. The body is clamped on $\Gamma_1 \times (0,T)$ and thus the displacement field vanishes there. A volume force of density $\boldsymbol{f}_0$ acts in $\Omega \times (0,T)$ and a surface traction of density $\boldsymbol{f}_2$ acts on $\Gamma_2 \times (0,T)$. In the reference configuration the body is in frictionless contact on Γ_3 with an obstacle, the so-called foundation. In the first problem we assume that contact is modelled with normal compliance. Under these assumptions, the classical formulation of the problem is the following.

Problem $\mathscr{P}_1$. *Find a displacement field $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathscr{S}^d$ such that*

$$\boldsymbol{\sigma}(t) = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}(s) - \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds \quad \text{in } \Omega \times (0,T), \tag{3.1}$$

$$\rho\ddot{\boldsymbol{u}} = \operatorname{Div}\boldsymbol{\sigma} + \boldsymbol{f}_0 \quad \text{in } \Omega \times (0,T), \tag{3.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1 \times (0,T), \tag{3.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{3.4}$$

$$-\sigma_\nu = p(u_\nu) \quad \text{on } \Gamma_3 \times (0,T), \tag{3.5}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \quad \text{on } \Gamma_3 \times (0,T), \tag{3.6}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0 \quad \text{in } \Omega. \tag{3.7}$$

Here (3.1) is the elastic-visco-plastic constitutive law introduced in Section 1, (3.2) represents the equation of motion in which ρ denotes the density of mass, (3.3) and (3.4) are the displacement and traction boundary conditions, respectively. Condition (3.5) represents the normal compliance condition in which σ_ν denotes the normal stress, u_ν is the normal displacement and p is a positive increasing function which vanishes for a negative argument; this condition shows that when there is separation between the body and the obstacle (i.e. when $u_\nu < 0$), then the reaction of the foundation vanishes (since $\sigma_\nu = 0$); also, when there is penetration (i.e. when $u_\nu \geq 0$), then the reaction of the foundation is towards the body (since $\sigma_\nu \leq 0$) and it is increasing with the penetration (since p is an increasing function). Condition (3.6) shows that the tangential shear, denoted $\boldsymbol{\sigma}_\tau$, vanishes on the contact surface, i.e. the process is frictionless. Finally, the functions $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$ in (3.7) denote the initial displacement and the initial velocity, respectively.

In the second problem we assume that contact is modelled with normal damped response and, therefore, the classical formulation of the problem is the following.

Problem $\mathscr{P}_2$. *Find a displacement field $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathscr{S}^d$ such that* (3.1)–(3.4),(3.6)–(3.7) *hold and, moreover,*

$$-\sigma_\nu = p(\dot{u}_\nu) \quad \text{on } \Gamma_3 \times (0,T). \tag{3.8}$$

The difference with respect problem $\mathscr{P}_1$ arise in the fact that in problem $\mathscr{P}_2$ we replace the normal compliance condition (3.5) with the normal damped response condition (3.8), which shows that now the normal stress depends on the normal velocity on the contact surface.

4. Variational formulation and main results

We now describe the assumptions on the data we consider in the study of the mechanical problems $\mathscr{P}_1$ and $\mathscr{P}_2$. Then we derive their variational foumulation and state our main existence and uniqueness results.

We assume that the operators $\mathscr{A}$, $\mathscr{E}$ and $\mathscr{G}$ satisfy the following conditions.

$$\begin{cases} \text{(a)}\mathscr{A}:\Omega\times\mathscr{S}^d\to\mathscr{S}^d.\\ \text{(b) There exists } L_{\mathscr{A}}>0 \text{ such that}\\ \quad \|\mathscr{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathscr{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\|\le L_{\mathscr{A}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|\\ \quad \forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathscr{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\ \text{(c) There exists } m_{\mathscr{A}}>0 \text{ such that}\\ \quad (\mathscr{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathscr{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2))\cdot(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)\ge m_{\mathscr{A}}\,\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|^2\\ \quad \forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathscr{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\ \text{(d) For each } \boldsymbol{\varepsilon}\in\mathscr{S}^d,\ \boldsymbol{x}\mapsto\mathscr{A}(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega.\\ \text{(e) The mapping } \boldsymbol{x}\mapsto\mathscr{A}(\boldsymbol{x},\boldsymbol{0}) \text{ belongs to } \mathcal{H}. \end{cases} \tag{4.1}$$

$$\begin{cases} \text{(a) } \mathscr{E}:\Omega\times\mathscr{S}^d\to\mathscr{S}^d.\\ \text{(b) There exists } L_{\mathscr{E}}>0 \text{ such that}\\ \quad \|\mathscr{E}(\boldsymbol{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1)-\mathscr{E}(\boldsymbol{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2)\|\le L_{\mathscr{E}}\left(\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\|+\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|\right)\\ \quad \forall\,\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathscr{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\ \text{(c) For each } \boldsymbol{\sigma},\boldsymbol{\varepsilon}\in\mathscr{S}^d,\ \boldsymbol{x}\mapsto\mathscr{E}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega.\\ \text{(d) The mapping } \boldsymbol{x}\mapsto\mathscr{E}(\boldsymbol{x},\boldsymbol{0},\boldsymbol{0}) \text{ belongs to } \mathcal{H}. \end{cases} \tag{4.2}$$

$$\begin{cases} \text{(a) } \mathscr{G}:\Omega\times\mathscr{S}^d\times\mathscr{S}^d\to\mathscr{S}^d.\\ \text{(b) There exists } L_{\mathscr{G}}>0 \text{ such that}\\ \quad \|\mathscr{G}(\boldsymbol{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1)-\mathscr{G}(\boldsymbol{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2)\|\le L_{\mathscr{G}}\left(\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\|+\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|\right)\\ \quad \forall\,\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathscr{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\ \text{(c) For each } \boldsymbol{\sigma},\boldsymbol{\varepsilon}\in\mathscr{S}^d,\ \boldsymbol{x}\mapsto\mathscr{G}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega.\\ \text{(d) The mapping } \boldsymbol{x}\mapsto\mathscr{G}(\boldsymbol{x},\boldsymbol{0},\boldsymbol{0}) \text{ belongs to } \mathcal{H}. \end{cases} \tag{4.3}$$

The contact function p satisfies

$$\begin{cases} \text{(a) } p:\Gamma_3\times\mathbb{R}\to\mathbb{R}.\\ \text{(b) There exists } L_p>0 \text{ such that}\\ \quad |p(\boldsymbol{x},r_1)-p(\boldsymbol{x},r_2)|\le L_p|r_1-r_2| \quad \forall r_1,\,r_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3.\\ \text{(c) } (p(\boldsymbol{x},r_1)-p(\boldsymbol{x},r_2))(r_1-r_2)\ge 0 \quad \forall r_1,\,r_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3.\\ \text{(d) For each } r\in\mathbb{R},\ \boldsymbol{x}\mapsto p(\boldsymbol{x},r) \text{ is measurable on } \Gamma_3.\\ \text{(e) } p(\boldsymbol{x},r)=0 \text{ for all } r<0 \text{ a.e. } \boldsymbol{x}\in\Gamma_3. \end{cases} \tag{4.4}$$

We also suppose that the mass density satisfies

$$\rho\in L^\infty(\Omega), \quad \text{there exists } \rho^*>0 \text{ such that } \rho(\boldsymbol{x})\ge\rho^* \text{ a.e. } \boldsymbol{x}\in\Omega, \tag{4.5}$$

the body forces and surface tractions have the regularity

$$\boldsymbol{f}_0\in L^2(0,T;L^2(\Omega)^d), \quad \boldsymbol{f}_2\in L^2(0,T;L^2(\Gamma_2)^d), \tag{4.6}$$

and the initial data satisfy

$$\boldsymbol{u}_0\in V, \quad \boldsymbol{v}_0\in L^2(\Omega)^d. \tag{4.7}$$

We turn now to the variational formulations of Problems $\mathscr{P}_1$ and $\mathscr{P}_2$. To this end we use a modified inner product on the Hilbert space $H=L^2(\Omega)^d$, given by

$$(\boldsymbol{u},\boldsymbol{v})_H=(\rho\,\boldsymbol{u},\boldsymbol{v})_{L^2(\Omega)^d} \quad \forall\,\boldsymbol{u},\,\boldsymbol{v}\in H, \tag{4.8}$$

that is, it is weighed with ρ, and we let $\|\cdot\|_H$ be the associated norm, i.e.,

$$\|\boldsymbol{v}\|_H=(\rho\,\boldsymbol{v},\boldsymbol{v})^{1/2}_{L^2(\Omega)^d} \quad \forall\,\boldsymbol{v}\in H. \tag{4.9}$$

It follows from assumption (4.5) that $\|\cdot\|_H$ and $\|\cdot\|_{L^2(\Omega)^d}$ are equivalent norms on H, and also the inclusion mapping of $(V,\|\cdot\|_V)$ into $(H,\|\cdot\|_H)$ is continuous and dense. We denote by V' the dual space of V. Identifying H with its own dual, we can write the Gelfand triple

$$V \subset H \subset V'.$$

We use the notation $\langle\cdot,\cdot\rangle_{V'\times V}$ to represent the duality pairing between V' and V and recall that

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle_{V'\times V} = (\boldsymbol{u},\boldsymbol{v})_H \quad \forall \boldsymbol{u}\in H,\, \boldsymbol{v}\in V \tag{4.10}$$

Assumptions (4.6) allow us, for almost any $t\in(0,T)$, to define $\boldsymbol{f}(t)\in V'$ by

$$\langle \boldsymbol{f}(t),\boldsymbol{v}\rangle_{V'\times V} = \int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{v}\,dx + \int_{\Gamma_2}\boldsymbol{f}_2(t)\cdot\boldsymbol{v}\,da \quad \forall \boldsymbol{v}\in V, \tag{4.11}$$

and note that

$$\boldsymbol{f}\in L^2(0,T;V'). \tag{4.12}$$

Finally, we consider the functional $j: V\times V\to\mathbb{R}$ defined by

$$j(\boldsymbol{u},\boldsymbol{v}) = \int_{\Gamma_3} p(u_\nu)v_\nu\,da, \quad \forall \boldsymbol{u},\,\boldsymbol{v}\in V \tag{4.13}$$

and we note that, by assumption (4.4), the integral in (4.13) is well defined.

We assume in what follows that $(\boldsymbol{u},\boldsymbol{\sigma})$ are smooth functions satisfying (3.2)–(3.6) and let $t\in[0,T]$. We take the dot product of equation (3.2) with $\boldsymbol{w}$ where $\boldsymbol{w}$ is an arbitrary element of V, integrate the result over Ω, and use Green's formula (2.1) to obtain

$$(\rho\,\ddot{\boldsymbol{u}}(t),\boldsymbol{w})_{L^2(\Omega)^d} + (\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} = \int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{w}\,dx + \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot\boldsymbol{w}\,da. \tag{4.14}$$

Applying the boundary conditions (3.4) and (3.6) and noting that $\boldsymbol{w}=\boldsymbol{0}$ on Γ_1, we have

$$\int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot\boldsymbol{w}\,da = \int_{\Gamma_2}\boldsymbol{f}_2(t)\cdot\boldsymbol{w}\,da + \int_{\Gamma_3}\sigma_\nu(t)\,w_\nu\,da. \tag{4.15}$$

Moreover, (3.5) combined with (4.13) lead to

$$\int_{\Gamma_3}\sigma_\nu(t)\,w_\nu\,da = -j(\boldsymbol{u}(t),\boldsymbol{w}). \tag{4.16}$$

We now use (4.14)–(4.16) and the equalities (4.8), (4.10) and (4.11) to find

$$\langle \ddot{\boldsymbol{u}}(t),\boldsymbol{w}\rangle_{V'\times V} + (\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + j(\boldsymbol{u}(t),\boldsymbol{w}) = \langle \boldsymbol{f}(t),\boldsymbol{w}\rangle_{V'\times V}. \tag{4.17}$$

Finally, we combine (3.1), (4.17), and (3.7) to derive the following variational formulation of Problem $\mathscr{P}_1$.

Problem $\mathscr{P}_1^V$. *Find a displacement field $\boldsymbol{u}:[0,T]\to V$ and a stress field $\boldsymbol{\sigma}:[0,T]\to\mathcal{H}$ such that, for a.e. $t\in(0,T)$,*

$$\boldsymbol{\sigma}(t) = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}(s) - \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s)),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds, \tag{4.18}$$

$$\langle \ddot{\boldsymbol{u}}(t),\boldsymbol{w}\rangle_{V'\times V} + (\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} + j(\boldsymbol{u}(t),\boldsymbol{w}) = \langle \boldsymbol{f}(t),\boldsymbol{w}\rangle_{V'\times V} \quad \forall \boldsymbol{w}\in V, \tag{4.19}$$

$$\boldsymbol{u}(0)=\boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0)=\boldsymbol{v}_0. \tag{4.20}$$

Using similar arguments we derive the following variational formulation of Problem $\mathscr{P}_2$.

Problem $\mathscr{P}_2^V$. *Find a displacement field $\boldsymbol{u} : [0,T] \to V$ and a stress field $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}$ such that* (4.18)*,* (4.20) *hold and, moreover,*

$$\langle \ddot{\boldsymbol{u}}(t), \boldsymbol{w}\rangle_{V'\times V} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} + j(\dot{\boldsymbol{u}}(t), \boldsymbol{w}) = \langle \boldsymbol{f}(t), \boldsymbol{w}\rangle_{V'\times V}, \tag{4.21}$$

for all $\boldsymbol{w} \in V$, a.e. $t \in (0,T)$.

Our main results that we state here and prove in the next section are the following.

Theorem 4.1. *Assume that conditions* (4.1)–(4.7) *hold. Then, Problem $\mathscr{P}_1^V$ has a unique solution. Moreover, the solution satisfies*

$$\boldsymbol{u} \in W^{1,2}(0,T;V) \cap C^1([0,T];H), \quad \ddot{\boldsymbol{u}} \in L^2(0,T;V'), \tag{4.22}$$

$$\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}), \quad \operatorname{Div} \boldsymbol{\sigma} \in L^2(0,T;V'). \tag{4.23}$$

Theorem 4.2. *Assume that conditions* (4.1)–(4.7) *hold. Then, Problem $\mathscr{P}_2^V$ has at least a solution. Moreover, the solution has the regularity expressed in* (4.22)–(4.23)*.*

We conclude by Theorems 4.1 and 4.2 that, under the assumptions (4.1)–(4.7), both the dynamic contact problem $\mathscr{P}_1$ and the dynamic contact problem $\mathscr{P}_2$ have a unique *weak solution* with regularity (4.22)–(4.23).

5. Proof of Theorems 4.1 and 4.2

We start with the proof of Theorem 4.1 which will be carried out in several steps. We assume in the rest of this section that (4.1)–(4.7) hold and c will denote a generic positive constant which may depend on Ω, Γ_1, Γ_2, Γ_3, $\mathscr{A}$, $\mathscr{E}$, $\mathscr{G}$, p and T, but does not depend on t, nor on the rest of the input data, and whose value may change from place to place. Let $\boldsymbol{\eta} \in L^2(0,T;V')$ be given. In the first step we consider the following variational problem :

Problem $\mathscr{P}_1^{\eta-disp}$. *Find a displacement field $\boldsymbol{u}_\eta : [0,T] \to V$ such that*

$$\begin{aligned} &\langle \ddot{\boldsymbol{u}}_\eta(t), \boldsymbol{w}\rangle_{V'\times V} + (\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} + \langle \boldsymbol{\eta}(t), \boldsymbol{w}\rangle_{V'\times V} \qquad (5.1)\\ &= \langle \boldsymbol{f}(t), \boldsymbol{w}\rangle_{V'\times V} \quad \forall\, \boldsymbol{w} \in V, \text{ a.e. } t \in (0,T),\end{aligned}$$

$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}_\eta(0) = \boldsymbol{v}_0. \tag{5.2}$$

In the study of Problem $\mathscr{P}_1^{\eta-disp}$ we have the following result.

Lemma 5.1. *There exists a unique solution to Problem $\mathscr{P}_1^{\eta-disp}$ and it has the regularity expressed in* (4.22)*. Moreover, if $\boldsymbol{u}_i$ represents the solution of Problem $\mathscr{P}_1^{\eta_i-disp}$ for $\boldsymbol{\eta}_i \in L^2(0,T;V')$, $i = 1,2$, then there exists $c > 0$ such that*

$$\int_0^t \|\dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s)\|_V^2\, ds \le c \int_0^t \|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_{V'}^2\, ds \quad \forall t \in [0,T]. \tag{5.3}$$

Proof. We define the operator $A : V \to V'$ by

$$\langle A\boldsymbol{v}, \boldsymbol{w}\rangle_{V'\times V} = (\mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} \quad \forall\, \boldsymbol{v},\, \boldsymbol{w} \in V. \tag{5.4}$$

It follows from (5.4) and (4.1)(b) that

$$\|A\boldsymbol{v} - A\boldsymbol{w}\|_{V'} \le L_{\mathscr{A}}\, \|\boldsymbol{v} - \boldsymbol{w}\|_V \quad \forall\, \boldsymbol{v},\, \boldsymbol{w} \in V, \tag{5.5}$$

which shows that $A : V \to V'$ is continuous and so is hemicontinuous. Now, by (5.4) and (4.1)(c), we find

$$\langle A\boldsymbol{v} - A\boldsymbol{w}, \boldsymbol{v} - \boldsymbol{w}\rangle_{V'\times V} \geq m_{\mathscr{A}} \, \|\boldsymbol{v} - \boldsymbol{w}\|_V^2 \quad \forall\, \boldsymbol{v},\, \boldsymbol{w} \in V, \tag{5.6}$$

i.e., that $A : V \to V'$ is a monotone operator. Choosing $\boldsymbol{w} = \boldsymbol{0}_V$ in (5.6) we obtain

$$\begin{aligned}\langle A\boldsymbol{v}, \boldsymbol{v}\rangle_{V'\times V} &\geq m_{\mathscr{A}} \|\boldsymbol{v}\|_V^2 - \|A\boldsymbol{0}_V\|_{V'} \|\boldsymbol{v}\|_V \\ &\geq \frac{1}{2}\, m_{\mathscr{A}} \, \|\boldsymbol{v}\|_V^2 - \frac{1}{2m_{\mathscr{A}}} \, \|A\boldsymbol{0}_V\|_{V'}^2 \quad \forall\, \boldsymbol{v} \in V.\end{aligned}$$

Thus, A satisfies condition (2.4) with $\omega = m_{\mathscr{A}}/2$ and $\alpha = -\|A\boldsymbol{0}_V\|_{V'}^2/(2m_{\mathscr{A}})$. Next, by (5.5) we deduce that

$$\|A\boldsymbol{v}\|_{V'} \leq L_{\mathscr{A}} \, \|\boldsymbol{v}\|_V + \|A\boldsymbol{0}_V\|_{V'} \quad \forall\, \boldsymbol{v} \in V.$$

This inequality implies that A satisfies condition (2.5). Finally, we recall that by (4.12), (4.7) we have $\boldsymbol{f} - \boldsymbol{\eta} \in L^2(0, T; V')$ and $\boldsymbol{v}_0 \in H$. It follows now from Theorem 2.1 that there exists a unique function $\boldsymbol{v}_\eta$ which satisfies

$$\boldsymbol{v}_\eta \in L^2(0, T; V) \cap C([0, T]; H), \;\; \dot{\boldsymbol{v}}_\eta \in L^2(0, T; V'), \tag{5.7}$$

$$\dot{\boldsymbol{v}}_\eta(t) + A\boldsymbol{v}_\eta(t) + \boldsymbol{\eta}(t) = \boldsymbol{f}(t) \quad \text{a.e. } t \in (0, T), \tag{5.8}$$

$$\boldsymbol{v}_\eta(0) = \boldsymbol{v}_0. \tag{5.9}$$

Let $\boldsymbol{u}_\eta : [0, T] \to V$ be the function defined by

$$\boldsymbol{u}_\eta(t) = \int_0^t \boldsymbol{v}_\eta(s)\, ds + \boldsymbol{u}_0 \quad \forall\, t \in [0, T]. \tag{5.10}$$

It follows from (5.4), (5.7)–(5.10) that $\boldsymbol{u}_\eta$ is a solution of the variational problem $\mathscr{P}_1^{\eta-disp}$ and it satisfies the regularity expressed in (4.22). This concludes the existence part of Lemma 5.1. The uniqueness part follows from the uniqueness of the solution of problem (5.7)–(5.9), guaranteed by Theorem 2.1.

Consider now $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in L^2(0, T; V')$ and denote $\boldsymbol{u}_i = \boldsymbol{u}_{\eta_i}$, $\boldsymbol{v}_i = \boldsymbol{v}_{\eta_i} = \dot{\boldsymbol{u}}_{\eta_i}$ for $i = 1, 2$. We use (5.1) to obtain

$$\begin{aligned}&\langle \dot{\boldsymbol{v}}_1 - \dot{\boldsymbol{v}}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2\rangle_{V'\times V} + (\mathscr{A}\, \boldsymbol{\varepsilon}(\boldsymbol{v}_1) - \mathscr{A}\, \boldsymbol{\varepsilon}(\boldsymbol{v}_2), \boldsymbol{\varepsilon}(\boldsymbol{v}_1) - \boldsymbol{\varepsilon}(\boldsymbol{v}_2))_{\mathcal{H}} \\ &+ \langle \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2\rangle_{V'\times V} = 0 \quad \text{a.e. on } \; (0, T).\end{aligned}$$

Let $t \in [0, T]$. We integrate the previous equality with respect to time and use the initial conditions $\boldsymbol{v}_1(0) = \boldsymbol{v}_2(0) = \boldsymbol{v}_0$ and the properties of the operator $\mathscr{A}$ to find

$$m_{\mathscr{A}} \int_0^t \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|_V^2 \, ds \leq -\int_0^t \langle \boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s), \boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\rangle_{V'\times V} \, ds.$$

Now,

$$\begin{aligned}&-\int_0^t \langle \boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s), \boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\rangle_{V'\times V} \, ds \\ &\leq \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{V'} \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|_V \, ds \\ &\leq \frac{1}{m_{\mathscr{A}}} \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{V'}^2 ds + \frac{m_{\mathscr{A}}}{4} \int_0^t \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|_V^2 \, ds.\end{aligned}$$

The previous two inequalities lead to

$$\int_0^t \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|_V^2\, ds \le c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{V'}^2 ds,$$

which implies (5.3). □

We use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 5.1 to construct the following Cauchy problem for the stress field.

Problem $\mathscr{P}_1^{\eta-st}$. *Find a stress field $\boldsymbol{\sigma}_\eta : [0,T] \to \mathcal{H}$ such that*

$$\boldsymbol{\sigma}_\eta(t) = \mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}_\eta(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)))\, ds \tag{5.11}$$

for all $t \in [0,T]$.

In the study of Problem $\mathscr{P}_V^{\eta-st}$ we have the following result.

Lemma 5.2. *There exists a unique solution of Problem $\mathscr{P}_1^{\eta-st}$ and it satisfies $\boldsymbol{\sigma}_\eta \in W^{1,2}(0,T;\mathcal{H})$. Moreover, if $\boldsymbol{\sigma}_i$ and $\boldsymbol{u}_i$ represent the solutions of problem $\mathscr{P}_1^{\eta_i-st}$ and $\mathscr{P}_1^{\eta_i-disp}$, respectively, for $\boldsymbol{\eta}_i \in L^2(0,T;V')$, $i = 1,2$, then there exists $c > 0$ such that*

$$\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} \le c \Big(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\, ds\Big) \tag{5.12}$$

for all $t \in [0,T]$.

Proof. Let $\Lambda_\eta : L^2(0,T;\mathcal{H}) \to L^2(0,T;\mathcal{H})$ be the operator given by

$$\Lambda_\eta \boldsymbol{\sigma}(t) = \mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)))\, ds \tag{5.13}$$

for all $\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H})$ and $t \in [0,T]$. For $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2 \in L^2(0,T;\mathcal{H})$ we use (5.13) and (4.3) to obtain

$$\|\Lambda_\eta\, \boldsymbol{\sigma}_1(t) - \Lambda_\eta\, \boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} \le L_{\mathscr{G}} \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}\, ds$$

for all $t \in [0,T]$. It follows from this inequality that for p large enough, a power Λ_η^p of the operator Λ_η is a contraction on the Banach space $L^2(0,T;V)$ and, therefore, there exists a unique element $\boldsymbol{\sigma}_\eta \in L^2(0,T;\mathcal{H})$ such that $\Lambda_\eta \boldsymbol{\sigma}_\eta = \boldsymbol{\sigma}_\eta$. Moreover, $\boldsymbol{\sigma}_\eta$ is the unique solution of Problem $\mathscr{P}_1^{\eta-st}$ and, using (5.11), the regularity of $\boldsymbol{u}_\eta$ and the properties of the operators $\mathscr{A}$, $\mathscr{E}$ and $\mathscr{G}$, it follows that $\boldsymbol{\sigma}_\eta \in W^{1,2}(0,T;\mathcal{H})$.

Consider now $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2 \in L^2(0,T;V')$ and, for $i = 1,2$, denote $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\boldsymbol{\sigma}_{\eta_i} = \boldsymbol{\sigma}_i$. We have

$$\boldsymbol{\sigma}_i(t) = \mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_i(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}_i(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_i(s)))\, ds \quad \forall t \in [0,T],$$

and, using the properties (4.2) and (4.3) of $\mathscr{E}$ and $\mathscr{G}$, we find

$$\begin{aligned}\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} \le c \Big(&\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V + \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}\, ds \\ &+ \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\, ds\Big) \quad \forall t \in [0,T].\end{aligned}$$

Using now a Gronwall argument in the previous inequality we deduce (5.12), which concludes the proof. □

We now introduce the operator $\Theta : L^2(0,T;V') \to L^2(0,T;V')$ which maps every element $\boldsymbol{\eta} \in L^2(0,T;V')$ to the element $\Theta\boldsymbol{\eta} \in L^2(0,T;V')$ defined by

$$\begin{aligned}\langle \Theta\boldsymbol{\eta}(t), \boldsymbol{w}\rangle_{V'\times V} = (\mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} + \Big(\int_0^t \mathscr{G}(\boldsymbol{\sigma}_\eta(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)))\,ds, \boldsymbol{\varepsilon}(\boldsymbol{w})\Big)_{\mathcal{H}} \\ + j(\boldsymbol{u}_\eta(t), \boldsymbol{w}) \quad \forall\, \boldsymbol{w} \in V, \quad \forall\, t \in [0,T].\end{aligned} \tag{5.14}$$

Here, for every $\boldsymbol{\eta} \in L^2(0,T;V')$, $\boldsymbol{u}_\eta$ and $\boldsymbol{\sigma}_\eta$ represent the displacement field and the stress field obtained in Lemmas 5.1 and 5.2, respectively. We have the following result.

Lemma 5.3. *The operator Θ has a unique fixed point $\boldsymbol{\eta}^* \in L^2(0,T,V')$.*

Proof. Let $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2 \in L^2(0,T;V')$, let $t \in [0,T]$ and denote $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\boldsymbol{\sigma}_{\eta_i} = \boldsymbol{\sigma}_i$, $i = 1,2$. We use (5.14), (4.2), (4.3) and elementary algebraic manipulations to obtain

$$\begin{aligned}&|\langle \Theta\boldsymbol{\eta}_1(t) - \Theta\boldsymbol{\eta}_2(t), \boldsymbol{w}\rangle_{V'\times V}| \\ &\le c\Big(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V + \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}\,ds \\ &\quad + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\,ds\Big)\|\boldsymbol{w}\|_V + |j(\boldsymbol{u}_1(t), \boldsymbol{w}) - j(\boldsymbol{u}_2(t), \boldsymbol{w})|.\end{aligned} \tag{5.15}$$

Now, it follows from (4.13) and (4.4) that

$$\begin{aligned}|j(\boldsymbol{u}_1(t), \boldsymbol{w}) - j(\boldsymbol{u}_2(t), \boldsymbol{w})| &\le c\int_{\Gamma_3} \|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|\,\|\boldsymbol{w}\|\,dx \\ &\le c\,\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{L^2(\Gamma_3)^d}\|\boldsymbol{w}\|_{L^2(\Gamma_3)^d};\end{aligned}$$

Using (2.3), we find

$$|j(\boldsymbol{u}_1(t), \boldsymbol{w}) - j(\boldsymbol{u}_2(t), \boldsymbol{w})| \le c\,\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V\|\boldsymbol{w}\|_V. \tag{5.16}$$

We substitute (5.16) in (5.15) and deduce that

$$\begin{aligned}\|\Theta\boldsymbol{\eta}_1(t) - \Theta\boldsymbol{\eta}_1(t)\|_{V'} \le c\Big(&\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\,ds \\ &+ \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}\,ds\Big).\end{aligned} \tag{5.17}$$

We use now (5.12) in (5.17) to obtain

$$\|\Theta\boldsymbol{\eta}_1(t) - \Theta\boldsymbol{\eta}_1(t)\|_{V'} \le c\Big(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\,ds\Big) \tag{5.18}$$

and, since $\boldsymbol{u}_1(0) = \boldsymbol{u}_2(0) = \boldsymbol{u}_0$, we have

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V \le \int_0^t \|\dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s)\|_V\,ds, \tag{5.19}$$

$$\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\,ds \le c\int_0^t \|\dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s)\|_V\,ds. \tag{5.20}$$

It follows from (5.18)–(5.20) that

$$\|\Theta\boldsymbol{\eta}_1(t) - \Theta\boldsymbol{\eta}_1(t)\|_{V'} \le c\int_0^t \|\dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s)\|_V\,ds,$$

which implies

$$\|\Theta\boldsymbol{\eta}_1(t)-\Theta\boldsymbol{\eta}_1(t)\|_{V'}^2 \le c\int_0^t \|\dot{\boldsymbol{u}}_1(s)-\dot{\boldsymbol{u}}_2(s)\|_V^2\,ds. \tag{5.21}$$

Lemma 5.3 is now a direct consequence of inequalities (5.21), (5.3) and Banach's fixed point theorem. □

We now have all the ingredients to prove Theorem 4.1.

Proof of Theorem 4.1. Let $\boldsymbol{\eta}^* \in L^2(0,T;V')$ be the fixed point of the operator Θ defined by (5.14) and denote

$$\boldsymbol{u}^* = \boldsymbol{u}_{\eta^*}, \quad \boldsymbol{\sigma}^* = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*) + \boldsymbol{\sigma}_{\eta^*}. \tag{5.22}$$

We prove that the couple $(\boldsymbol{u}^*,\boldsymbol{\sigma}^*)$ satisfies (4.18)–(4.20), (4.22) and (4.23). Indeed, we write (5.11) for $\boldsymbol{\eta}=\boldsymbol{\eta}^*$ and use (5.22) to obtain that (4.18) is satisfied. Then we use (5.1) for $\boldsymbol{\eta}=\boldsymbol{\eta}^*$ to find

$$\begin{aligned}&\langle \ddot{\boldsymbol{u}}^*(t),\boldsymbol{w}\rangle_{V'\times V} + (\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*(t)),\boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} + \langle \boldsymbol{\eta}^*(t),\boldsymbol{w}\rangle_{V'\times V}\\ &= \langle \boldsymbol{f}(t),\boldsymbol{w}\rangle_{V'\times V} \quad \forall\,\boldsymbol{w}\in V,\ \text{a.e. } t\in(0,T).\end{aligned} \tag{5.23}$$

Equality $\Theta\boldsymbol{\eta}^*=\boldsymbol{\eta}^*$ combined with (5.14) and (5.22) shows that

$$\begin{aligned}\langle \boldsymbol{\eta}^*(t),\boldsymbol{w}\rangle_{V'\times V} &= (\mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}^*(t)),\boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}}\\ &\quad + \Big(\int_0^t \mathscr{G}(\boldsymbol{\sigma}^*(s)-\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*(s)),\boldsymbol{\varepsilon}(\boldsymbol{u}^*(s)))\,ds,\boldsymbol{\varepsilon}(\boldsymbol{w})\Big)_{\mathcal{H}}\\ &\quad + j(\boldsymbol{u}^*(t),\boldsymbol{w}) \quad \forall\,\boldsymbol{w}\in V,\ t\in[0,T].\end{aligned} \tag{5.24}$$

We now substitute (5.24) in (5.23) and use (4.18), to see that $(\boldsymbol{u}^*,\boldsymbol{\sigma}^*)$ satisfies (4.19). Next, (4.20) and (4.22) follow from Lemma 5.1 and the regularity $\boldsymbol{\sigma}^*\in L^2(0,T;\mathcal{H})$ follows from Lemmas 5.1, 5.2 and the second equality in (5.22). Finally (4.19) implies that

$$\rho\ddot{\boldsymbol{u}}^*(t) = \operatorname{Div}\boldsymbol{\sigma}^*(t) + \boldsymbol{f}_0(t) \quad \text{in } V', \quad \text{a.e. } t\in(0,T),$$

and therefore by (4.6) we find that $\operatorname{Div}\boldsymbol{\sigma}^*\in L^2(0,T;V')$. We deduce that (4.23) holds which concludes the existence part of the theorem. The uniqueness part is a consequence of the uniqueness of the fixed point of the operator Θ defined by (5.14). □

The proof of Theorem 4.2 is similar to that of Theorem 4.1 and is carried out in several steps. Since the modifications are straightforward, we omit the details.

Proof of Theorem 4.2. The steps of the proof are the following.

(i) For every $\boldsymbol{\eta}\in L^2(0,T;V')$ we prove that there exists a unique function $\boldsymbol{u}_\eta$ with regularity (4.22) such that

$$\begin{aligned}&\langle \ddot{\boldsymbol{u}}_\eta(t),\boldsymbol{w}\rangle_{V'\times V} + (\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t)),\boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} + j(\dot{\boldsymbol{u}}_\eta(t)),\boldsymbol{w}) + \langle \boldsymbol{\eta}(t),\boldsymbol{w}\rangle_{V'\times V}\\ &= \langle \boldsymbol{f}(t),\boldsymbol{w}\rangle_{V'\times V} \quad \forall\,\boldsymbol{w}\in V,\ \text{a.e. } t\in(0,T),\end{aligned} \tag{5.25}$$

$$\boldsymbol{u}_\eta(0)=\boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}_\eta(0)=\boldsymbol{v}_0. \tag{5.26}$$

To prove that this holds, we define the operator $A: V\to V'$ by

$$\langle A\boldsymbol{v},\boldsymbol{w}\rangle_{V'\times V} = (\mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{v}),\boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} + j(\boldsymbol{v},\boldsymbol{w}) \quad \forall\,\boldsymbol{v},\,\boldsymbol{w}\in V. \tag{5.27}$$

We prove that A satisfies conditions (2.4) and (2.5), then we use again Theorem 2.1 and proceed like in the proof of Lemma 5.1.

Moreover, we use estimates similar to those in the prof of Lemma 5.1 to see that, if $\boldsymbol{u}_i$ represents the solution of problem (5.25)–(5.26) for $\boldsymbol{\eta}_i \in L^2(0,T;V')$, $i=1,2$, then there exists $c>0$ such that (5.3) holds.

(ii) We use the displacement field $\boldsymbol{u}_\eta$ obtained in step i) and Lemma 5.2 to prove that there exists a unique function $\boldsymbol{\sigma}_\eta \in W^{1,2}(0,T;\mathcal{H})$ which satisfies (5.11) for all $t \in [0,T]$. Moreover, if $\boldsymbol{\sigma}_i$ and $\boldsymbol{u}_i$ represent the solutions obtained above for $\boldsymbol{\eta}_i \in L^2([0,T];V')$, $i=1,2$, then there exists $c>0$ such that (5.12) holds.

(iii) We now introduce the operator $\Theta : L^2(0,T;V') \to L^2(0,T;V')$ which maps every element $\boldsymbol{\eta} \in L^2(0,T;V')$ to the element $\Theta\boldsymbol{\eta} \in L^2(0,T;V')$ defined by

$$\langle \Theta\boldsymbol{\eta}(t), \boldsymbol{w}\rangle_{V'\times V} = (\mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} + (\int_0^t \mathscr{G}(\boldsymbol{\sigma}_\eta(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)))\,ds, \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} \tag{5.28}$$

for all $\boldsymbol{w} \in V$, $t \in [0,T]$. Here, for every $\boldsymbol{\eta} \in L^2(0,T;V')$, $\boldsymbol{u}_\eta$ and $\boldsymbol{\sigma}_\eta$ represent the displacement field and the stress field obtained in steps i) and ii) respectively. We use (5.12) and estimates similar to those used in Lemma 5.3 to prove that the operator Θ satisfies (5.21). It follows now from (5.3) and Banach's fixed point theorem and that the operator Θ has a unique fixed point $\boldsymbol{\eta}^* \in L^2(0,T,V')$.

(iv) Let $\boldsymbol{\eta}^* \in L^2(0,T;V')$ be the fixed point of the operator Θ defined by (5.28) and denote

$$\boldsymbol{u}^* = \boldsymbol{u}_{\eta^*}, \quad \boldsymbol{\sigma}^* = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*) + \boldsymbol{\sigma}_{\eta^*}. \tag{5.29}$$

We use equality $\Theta\boldsymbol{\eta}^* = \boldsymbol{\eta}^*$, (5.29) and the definition (5.28) of the operator Θ to prove that the couple $(\boldsymbol{u}^*, \boldsymbol{\sigma}^*)$ is a solution of Problem $\mathscr{P}_2^V$ and it satisfies (4.22)–(4.23). This concludes the existence part of the theorem. The uniqueness follows from the uniqueness of the fixed point of the operator Θ defined by (5.28), obtained in step iii). □

6. Continuous dependence results

In this section we study the dependence of the solution of the problem $\mathscr{P}_1^V$ and $\mathscr{P}_2^V$ with respect to a perturbation of the contact conditions. To avoid repetitions we restrict ourselves to the study of Problem $\mathscr{P}_1^V$ and we note that a result similar to that in Theorem 6.1 below can be obtained in the study of Problem $\mathscr{P}_2^V$. We suppose in what follows that (4.1)–(4.7) hold and denote by $(\boldsymbol{u}, \boldsymbol{\sigma})$ the solution of Problem $\mathscr{P}_1^V$ obtained in Theorem 4.1. Also, for all $\alpha > 0$ we denote by p^α a perturbation of p, which satisfies (4.4) with L_p replaced by L_p^α. We introduce the functional j^α defined by (4.13) replacing p with p^α and we consider the following variational problem.

Problem $\mathscr{P}_1^{V^\alpha}$. *Find a displacement field $\boldsymbol{u}^\alpha : [0,T] \to V$ and a stress field $\boldsymbol{\sigma}^\alpha : [0,T] \to \mathcal{H}$ such that, for a.e. $t \in (0,T)$,*

$$\boldsymbol{\sigma}^\alpha(t) = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha(t)) + \mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}^\alpha(s) - \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha(s)))\,ds, \tag{6.1}$$

$$\langle \ddot{\boldsymbol{u}}^\alpha(t), \boldsymbol{w}\rangle_{V'\times V} + (\boldsymbol{\sigma}^\alpha(t), \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} + j^\alpha(\boldsymbol{u}^\alpha(t), \boldsymbol{w}) = \langle \boldsymbol{f}(t), \boldsymbol{w}\rangle_{V'\times V} \quad \forall \boldsymbol{w} \in V, \tag{6.2}$$

$$\boldsymbol{u}^\alpha(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}^\alpha(0) = \boldsymbol{v}_0. \tag{6.3}$$

We deduce from the Theorem 4.1 that, for every $\alpha > 0$, problem $\mathscr{P}_1^{V^\alpha}$ has a unique solution $(\boldsymbol{u}^\alpha, \boldsymbol{\sigma}^\alpha)$ which satisfies (4.22)–(4.23). Assume that the contact function satisfies the following hypothesis :

$$\left\{\begin{array}{l}\text{There exist } \beta\in\mathbb{R}_+ \text{ and } \theta:]0,+\infty[\to[0,+\infty[\ \text{ such that}\\ \text{(a) } |p^\alpha(\boldsymbol{x},r)-p(\boldsymbol{x},r)|\le\theta(\alpha)(|r|+\beta)\quad\forall\alpha>0,\quad r\in\mathbb{R},\ \text{p.p. } \boldsymbol{x}\in\Gamma_3.\\ \text{(b) } \lim_{\alpha\to0}\theta(\alpha)=0.\end{array}\right. \tag{6.4}$$

$$\text{There exists } \ L_0>0 \ \text{ such that } \ L^\alpha\le L_0 \ \text{ for all } \ \alpha>0. \tag{6.5}$$

Under these hypotheses, we have the following convergence result.

Theorem 6.1. *The solution $(\boldsymbol{u}^\alpha,\boldsymbol{\sigma}^\alpha)$ of the problem $\mathscr{P}_1^{V^\alpha}$ converges to the solution $(\boldsymbol{u},\boldsymbol{\sigma})$ of the problem $\mathscr{P}_1^V$, i.e.*

$$\boldsymbol{u}^\alpha\to\boldsymbol{u}\quad\text{in}\quad W^{1,2}(0,T;V),\quad \boldsymbol{\sigma}^\alpha\to\boldsymbol{\sigma}\quad\text{in}\quad L^2(0,T;\mathcal{H})\quad\text{as}\quad\alpha\to0. \tag{6.6}$$

In addition to the interest in this convergence result from the asymptotic analysis point of view, it is important from mechanical point of view since it shows that small perturbations on the contact conditions lead to small perturbations of the weak solution of the dynamic contact problem $\mathscr{P}_1$.

Proof. Let $\alpha>0$. Everywhere below c denotes a positive constant which may depend on the data and on the solution $(\boldsymbol{u},\boldsymbol{\sigma})$ but does not depend on α, nor on the time variable, and whose value may change from line to line. Using (4.19) and (6.2) we obtain

$$\begin{aligned}&(\ddot{\boldsymbol{u}}^\alpha(t)-\ddot{\boldsymbol{u}}(t),\dot{\boldsymbol{u}}^\alpha(t)-\dot{\boldsymbol{u}}(t))_{V'\times V}+(\boldsymbol{\sigma}^\alpha(t)-\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha(t))-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_{\mathcal{H}}\\&+j^\alpha(\boldsymbol{u}^\alpha(t),\dot{\boldsymbol{u}}^\alpha(t)-\dot{\boldsymbol{u}}(t))-j(\boldsymbol{u}(t),\dot{\boldsymbol{u}}^\alpha(t)-\dot{\boldsymbol{u}}(t))\\&=0\quad\text{a.e. } t\in(0,T).\end{aligned} \tag{6.7}$$

We define

$$\boldsymbol{\sigma}^{\alpha R}(t)=\boldsymbol{\sigma}^\alpha(t)-\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha(t)),\quad \boldsymbol{\sigma}^R(t)=\boldsymbol{\sigma}(t)-\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) \tag{6.8}$$

and note that (6.1) and (4.18) yield

$$\begin{aligned}\boldsymbol{\sigma}^{\alpha R}(t)&=\mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha(t))+\int_0^t\mathscr{G}(\boldsymbol{\sigma}^{\alpha R}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha(s)))ds,\\ \boldsymbol{\sigma}^R(t)&=\mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))+\int_0^t\mathscr{G}(\boldsymbol{\sigma}^R(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))ds\end{aligned} \tag{6.9}$$

for all $t\in[0,T]$. We combine (6.7) and (6.8) to obtain

$$\begin{aligned}&(\ddot{\boldsymbol{u}}^\alpha(t)-\ddot{\boldsymbol{u}}(t),\dot{\boldsymbol{u}}^\alpha(t)-\dot{\boldsymbol{u}}(t))_{V'\times V}\\&+(\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha(t))-\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)),\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha(t))-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_{\mathcal{H}}\\&=-(\boldsymbol{\sigma}^{\alpha R}(t)-\boldsymbol{\sigma}^R(t),\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha(t))-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_{\mathcal{H}}\\&\quad+j(\boldsymbol{u}(t),\dot{\boldsymbol{u}}^\alpha(t)-\dot{\boldsymbol{u}}(t))-j^\alpha(\boldsymbol{u}^\alpha(t),\dot{\boldsymbol{u}}^\alpha(t)-\dot{\boldsymbol{u}}(t))\quad\text{a.e. } t\in(0,T).\end{aligned} \tag{6.10}$$

It follows from (4.1) that

$$(\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha(t))-\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)),\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha(t))-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_{\mathcal{H}}\ge m_{\mathscr{A}}\,\|\dot{\boldsymbol{u}}^\alpha(t)-\dot{\boldsymbol{u}}(t)\|_V^2 \tag{6.11}$$

a.e. $t\in(0,T)$. Using (6.9),

$$\begin{aligned}\boldsymbol{\sigma}^{\alpha R}(t)-\boldsymbol{\sigma}^R(t)&=\mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha(t))-\mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))+\int_0^t\mathscr{G}(\boldsymbol{\sigma}^{\alpha R}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha(s)))ds\\&\quad-\int_0^t\mathscr{G}(\boldsymbol{\sigma}^R(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))ds\quad\forall t\in[0,T].\end{aligned} \tag{6.12}$$

We use now (6.12), (4.2) and (4.3) to obtain

$$\|\boldsymbol{\sigma}^{\alpha R}(t)-\boldsymbol{\sigma}^{R}(t)\|_{\mathcal{H}} \le c\Big(\|\boldsymbol{u}^{\alpha}(t) -\boldsymbol{u}(t)\|_V+\int_0^t \|\boldsymbol{\sigma}^{\alpha R}(s)-\boldsymbol{\sigma}^{R}(s)\|_{\mathcal{H}} ds + \int_0^t \|\boldsymbol{u}^{\alpha}(s)-\boldsymbol{u}(s)\|_V\,ds\Big) \quad \forall t\in[0,T].$$

After a Gronwall argument we deduce

$$\|\boldsymbol{\sigma}^{\alpha R}(t)-\boldsymbol{\sigma}^{R}(t)\|_{\mathcal{H}} \le c\Big(\|\boldsymbol{u}^{\alpha}(t)-\boldsymbol{u}(t)\|_V+\int_0^t \|\boldsymbol{u}^{\alpha}(s)-\boldsymbol{u}(s)\|_V\,ds\Big) \tag{6.13}$$

for all $t\in[0,T]$. The above inequality shows that

$$\begin{aligned}&-(\boldsymbol{\sigma}^{\alpha R}(t)-\boldsymbol{\sigma}^{R}(t),\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{\alpha}(t))-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_{\mathcal{H}}\\ &\le c\Big(\|\boldsymbol{u}^{\alpha}(t)-\boldsymbol{u}(t)\|_V+\int_0^t \|\boldsymbol{u}^{\alpha}(s)-\boldsymbol{u}(s)\|_V\,ds\Big)\|\dot{\boldsymbol{u}}^{\alpha}(t)-\dot{\boldsymbol{u}}(t)\|_V \quad \text{a.e. } t\in(0,T).\end{aligned} \tag{6.14}$$

Note that from the definition of the functionals j and j^{α} it follows that

$$\begin{aligned}&j(\boldsymbol{u}(t),\dot{\boldsymbol{u}}^{\alpha}(t)-\dot{\boldsymbol{u}}(t))-j^{\alpha}(\boldsymbol{u}^{\alpha}(t),\dot{\boldsymbol{u}}^{\alpha}(t)-\dot{\boldsymbol{u}}(t))\\ &=\int_{\Gamma_3}(p(u_\nu(t))-p^{\alpha}(u_\nu(t)))(\dot{u}^{\alpha}_\nu(t))-\dot{u}_\nu(t))\,da\\ &\quad+\int_{\Gamma_3}(p^{\alpha}(u_\nu(t))-p^{\alpha}(u^{\alpha}_\nu(t)))(\dot{u}^{\alpha}_\nu(t))-\dot{u}_\nu(t))\,da \quad \text{a.e. } t\in(0,T).\end{aligned}$$

Using (6.4), (6.5) and (2.3) we deduce that

$$\begin{aligned}&j(\boldsymbol{u}(t),\dot{\boldsymbol{u}}^{\alpha}(t)-\dot{\boldsymbol{u}}(t))-j^{\alpha}(\boldsymbol{u}^{\alpha}(t),\dot{\boldsymbol{u}}^{\alpha}(t)-\dot{\boldsymbol{u}}(t))\\ &\le c\left(\theta(\alpha)+\|\boldsymbol{u}^{\alpha}(t)-\boldsymbol{u}(t)\|_V\right)\|\dot{\boldsymbol{u}}^{\alpha}(t)-\dot{\boldsymbol{u}}(t)\|_V \quad \text{a.e. } t\in(0,T).\end{aligned} \tag{6.15}$$

We use now (6.10), (6.11), (6.14) and (6.15) to obtain

$$\begin{aligned}&(\ddot{\boldsymbol{u}}^{\alpha}(t)-\ddot{\boldsymbol{u}}(t),\dot{\boldsymbol{u}}^{\alpha}(t)-\dot{\boldsymbol{u}}(t))_{V'\times V}+m_{\mathscr{A}}\,\|\dot{\boldsymbol{u}}^{\alpha}(t)-\dot{\boldsymbol{u}}(t)\|_V^2\\ &\le c\Big(\theta(\alpha)+\|\boldsymbol{u}^{\alpha}(t)-\boldsymbol{u}(t)\|_V+\int_0^t \|\boldsymbol{u}^{\alpha}(s)-\boldsymbol{u}(s)\|_V ds\Big)\|\dot{\boldsymbol{u}}^{\alpha}(t)-\dot{\boldsymbol{u}}(t)\|_V\end{aligned} \tag{6.16}$$

a.e. $t\in(0,T)$. Using the inequality

$$ab\le \frac{1}{2m_{\mathscr{A}}}\,a^2+\frac{m_{\mathscr{A}}}{2}\,b^2,$$

after some algebra we find that

$$\begin{aligned}&(\ddot{\boldsymbol{u}}^{\alpha}(t)-\ddot{\boldsymbol{u}}(t),\dot{\boldsymbol{u}}^{\alpha}(t)-\dot{\boldsymbol{u}}(t))_{V'\times V}+\frac{m_{\mathscr{A}}}{2}\,\|\dot{\boldsymbol{u}}^{\alpha}(t)-\dot{\boldsymbol{u}}(t)\|_V^2\\ &\le c\Big(\theta^2(\alpha)+\|\boldsymbol{u}^{\alpha}(t)-\boldsymbol{u}(t)\|_V^2+\int_0^t \|\boldsymbol{u}^{\alpha}(s)-\boldsymbol{u}(s)\|_V^2 ds\Big) \quad \text{a.e. } t\in(0,T).\end{aligned}$$

We integrate the previous inequality on $[0,s]$ and use the initial conditions $\dot{\boldsymbol{u}}^{\alpha}(0)=\dot{\boldsymbol{u}}(0)=\boldsymbol{v}_0$ to find that

$$\frac{m_{\mathscr{A}}}{2}\int_0^s \|\dot{\boldsymbol{u}}^{\alpha}(t)-\dot{\boldsymbol{u}}(t)\|_V^2 \le c\Big(\theta^2(\alpha)+\int_0^s \|\boldsymbol{u}^{\alpha}(t)-\boldsymbol{u}(t)\|_V^2 ds\Big) \quad \forall s\in[0,T]. \tag{6.17}$$

We use now (4.20) and (6.3) to see that

$$\|\boldsymbol{u}^{\alpha}(s)-\boldsymbol{u}(s)\|_V^2 \le c\int_0^s \|\dot{\boldsymbol{u}}^{\alpha}(t)-\dot{\boldsymbol{u}}(t)\|_V^2\,dt \quad \forall s\in[0,T]. \tag{6.18}$$

We substitute (6.17) in (6.18) then we use again the Gronwall inequality to find that

$$\|\boldsymbol{u}^{\alpha}(s)-\boldsymbol{u}(s)\|_V^2 \le c\,\theta^2(\alpha) \quad \forall s\in[0,T]. \tag{6.19}$$

Using now (6.17) and (6.19) it follows that

$$\int_0^s \|\dot{\boldsymbol{u}}^{\alpha}(t)-\dot{\boldsymbol{u}}(t)\|_V^2\,ds \le c\,\theta^2(\alpha) \quad \forall s\in[0,T]. \tag{6.20}$$

We combine now (6.19), (6.20) and use assumption (6.4)(b) to see that

$$\boldsymbol{u}^{\alpha}\to\boldsymbol{u} \quad \text{in} \quad W^{1,2}(0,T;V) \quad \text{as} \quad \alpha\to 0. \tag{6.21}$$

It follows from (6.8) that

$$\boldsymbol{\sigma}^{\alpha}(t)-\boldsymbol{\sigma}(t)=\boldsymbol{\sigma}^{\alpha R}(t)-\boldsymbol{\sigma}^{R}(t)+\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{\alpha}(t))-\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) \quad \text{a.e. } t\in(0,T).$$

Using this inequality, (6.13), the properties (4.1) of the operator $\mathscr{A}$ and (6.21) in obtain

$$\boldsymbol{\sigma}^{\alpha}\to\boldsymbol{\sigma} \quad \text{in } L^2(0,T;\mathcal{H}) \quad \text{as } \alpha\to 0. \tag{6.22}$$

Theorem 6.1 is now a consequence of (6.21), (6.22). □

References

[1] A. Amassad and C. Fabre; Existence for viscoplastic contact with Coulomb friction problems, *Int. J. Math. Math. Sci.* **32** (2002), 411–437.

[2] A. Amassad, C. Fabre and M. Sofonea; A quasistatic viscoplastic contact problem with normal compliance and friction, *IMA Journal of Applied Mathematics* **69** (2004), 463–482.

[3] L.-E. Andersson; A quasistatic frictional problem with normal compliance, *Nonlinear Analysis TMA* **16** (1991), 407–428.

[4] L.-E. Andersson; A global existence result for a quasistatic contact problem with friction, *Advances in Mathemtical Sciences and Applications* **5** (1995), 249–286.

[5] V. Barbu; *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Editura Academiei, Bucharest-Noordhoff, Leyden, 1976.

[6] N. Cristescu and I. Suliciu; *Viscoplasticity*, Martinus Nijhoff Publishers, Editura Tehnica, Bucharest, 1982.

[7] C. Eck, J. Jarušek and M. Krbeč; *Unilateral Contact Problems: Variational Methods and Existence Theorems*, Pure and Applied Mathematics **270**, Chapman/CRC Press, New York, 2005.

[8] J. R. Fernández-García, M. Sofonea and J. M. Viaño; A frictionless contact problem for elastic-viscoplastic materials with normal compliance, *Numerische Mathemätik* **90** (2002), 689–719.

[9] W. Han and M. Sofonea; *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, Studies in Advanced Mathematics **30**, American Mathematical Society, Providence, RI—International Press, Somerville, MA, 2002.

[10] I. R. Ionescu and M. Sofonea; *Functional and Numerical Methods in Viscoplasticity*, Oxford University Press, Oxford, 1994.

[11] J. Jarušek; Dynamic contact problems with given friction for viscoelastic bodies, *Czechoslovak Mathematical Journal* **46** (1996), 475–487.

[12] J. Jarušek and C. Eck; Dynamic contact problems with small Coulomb friction for viscoelastic bodies. Existence of solutions, *Mathematical Models and Methods in Applied Sciences* **9** (1999), 11–34.

[13] N. Kikuchi and J. T. Oden; *Contact Problems in Elasticity: A Study of VariationalInequalities and Finite Element Methods*, SIAM, Philadelphia, 1988.

[14] A. Klarbring, A. Mikelič and M. Shillor; Frictional contact problems with normal compliance, *Int. J. Engng. Sci.* **26** (1988), 811–832.

[15] K. L. Kuttler; Dynamic friction contact problem with general normal and friction laws, *Nonlinear Analysis* **28** (1997), 559–575.

[16] K. L. Kuttler and M. Shillor; Set-valued pseudomonotone maps and degenerate evolution inclusions, *Communications in Contemporary Mathematics* **1** (1999), 87–123.

[17] S. Migórski; Dynamic hemivariational inequality modeling viscoelastic contact problem with normal damped response and friction, *Applicable Analysis* **84** (2005), 669–699.

[18] S. Migórski and A. Ochal; A unified approach to dynamic contact problems in viscoelasticity, *J. Elasticity* **83** (2006), 247–275.

[19] J. A. C. Martins and J. T. Oden; Existence and uniqueness results for dynamic contact problems with nonlinear normal and friction interface laws, *Nonlinear Analysis TMA* **11** (1987), 407–428.

[20] M. Rochdi, M. Shillor and M. Sofonea; A quasistatic viscoelastic contact problem with normal compliance and friction, *Journal of Elasticity* **51** (1998), 105–126.

[21] M. Rochdi, M. Shillor and M. Sofonea; A quasistatic contact problem with directional friction and damped response, *Applicable Analysis* **68** (1998), 409–422.

[22] M. Shillor, M. Sofonea and J. J. Telega; *Models and Analysis of Quasistatic Contact*, Lecture Notes in Physics **655**, Springer, Berlin, 2004.

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