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BEHAVIOR OF IMPULSIVE FUZZY CELLULAR NEURAL NETWORKS WITH DISTRIBUTED DELAYS

KELIN LI, ZUOAN LI, QIANKUN SONG

Abstract. In this paper, we investigate a generalized model of fuzzy cellular neural networks with distributed delays and impulses. By employing the theory of topological degree, M-matrix and Lypunov functional, we find sufficient conditions for the existence, uniqueness and global exponential stability of both the equilibrium point and the periodic solution. Two examples are given to illustrate the results obtained here.

1. Introduction

Since cellular neural networks (CNN) was introduced by Chua and Yang in [7, 8], many researchers have done extensive works on this subject due to their applications in classification of patterns, associative memories, image processing, quadratic optimization, and other areas, e.g., [2, 5, 6, 18, 19, 25]. However, in mathematical modelling of real world problems, we encounter inconveniences, namely, the complexity and the uncertainty or vagueness. In order to take vagueness into consideration, fuzzy theory is considered as a suitable setting. Based on traditional CNN, Yang and Yang proposed the fuzzy cellular neural networks (FCNN) [22, 23], which integrates fuzzy logic into the structure of the traditional CNN and maintains local connectedness among cells. Unlike previous CNN structures, FCNN has fuzzy logic between its template input and/or output besides the sum of product operation. FCNN is very useful paradigm for image processing problems, which is a cornerstone in image processing and pattern recognition. In such applications, it is of prime importance to ensure that the designed FCNN be stable. In [22, 23], the authors have obtained some conditions for the existence and the global stability of the equilibrium point of FCNN without delays. In [16], Liu and Tang have considered FCNN with either constant delays or time-varying delays, several sufficient conditions have been obtained to ensure the existence and uniqueness of the equilibrium point and its global exponential stability. Yuan, Cao and Deng have given several novel criteria of exponential stability and periodic solutions for FCNN with time-varying delays [24]. Recently, Huang has considered the stability of FCNN with diffusion terms and time-varying delay [14], at the same time, Huang has investigated the exponential stability of FCNN with distributed delay [13].

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However, besides delay effect, impulsive effect likewise exists in a wide variety of evolutionary processes in which states are changed abruptly at certain moments of time, involving such fields as medicine and biology, economics, mechanics, electronics and telecommunications, etc. Many interesting results on impulsive effect have been gained, e.g., Refs. [1, 3, 10, 11, 15, 17, 20, 21]. As artificial electronic systems, neural networks such as CNN, bidirectional neural networks and recurrent neural networks often are subject to impulsive perturbations which can affect dynamical behaviors of the systems just as time delays. Therefore, it is necessary to consider both impulsive effect and delay effect on the stability of neural networks.

Motivated by the above discussions, in this paper, on the basis of the structure of FCNN, we consider a class of impulsive fuzzy neural networks with distributed delays described by the following system of integro-differential equations:

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} \tilde{a}_{ij} u_j(t) + I_i(t) \\ &\quad + \bigwedge_{j=1}^{n} b_{ij} \int_{-\infty}^{t} K_{ij}(t-s) f_j(x_j(s)) ds + \bigvee_{j=1}^{n} \tilde{b}_{ij} \int_{-\infty}^{t} K_{ij}(t-s) f_j(x_j(s)) ds \\ &\quad + \bigwedge_{j=1}^{n} T_{ij} u_j(t) + \bigvee_{j=1}^{n} H_{ij} u_j(t), \quad t \neq t_k, \\ \Delta x_i(t_k) &= x_i(t_k^+) - x_i(t_k^-) = \Delta_k(x_i(t_k)), \quad t = t_k, \end{aligned} \tag{1.1}$$

for $i = 1, 2, \dots, n$. Where the fixed times t_k satisfy $t_1 < t_2 < \dots$, $\lim_{k\to\infty} t_k = \infty$. The first part (called continuous part) of model (1.1) describes the continuous processes of FCNN. n corresponds to the number of units in the neural network; x_i corresponds to the state variable; $f_j(x_j(t))$ denotes the activation function of the jth neuron; u_i and $I_i(t)$ denote input and bias of the ith neuron, respectively. d_i represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the networks and external inputs; a_{ij} and $\tilde{a}_{ij}$ are elements of feedback template and feedforward template; b_{ij}, $\tilde{b}_{ij}$ are elements of the distributed delay fuzzy feedback MIN template, the distributed delay fuzzy feedback MAX template, T_{ij} and H_{ij} are elements of fuzzy feedforward MIN template and fuzzy feedforward MAX template, respectively; K_{ij} corresponds to the delay kernel. $\bigwedge$ and $\bigvee$ denote the fuzzy AND and fuzzy OR operation, respectively. The second part (called discrete part) of model (1.1) describes that the evolution processes experience abrupt change of states at the moments of time t_k (called impulsive moments). $\Delta x_i(t_k)$ represents impulsive perturbations of the ith unit at time t_k, and Δ_k denotes the impulsive operator at time t_k for $k = 1, 2, \dots$.

To the best of our knowledge, few authors has considered dynamical behaviors of impulsive fuzzy neural networks with distributed delays. This paper studies the existence, uniqueness and global exponential stability of both the equilibrium point and the periodic solution for impulsive fuzzy neural networks with distributed delays. Several sufficient conditions ensuring the existence, uniqueness and global exponential stability of both the equilibrium point and the periodic solution for impulsive fuzzy neural networks with distributed delays will be established for the system (1.1).

The remainder part of this paper is organized as follows. some notations and preliminaries are given in section 2. In section 3, several sufficient conditions will be established ensuring model (1.1) to the existence, uniqueness and global exponential stability of equilibrium point. The existence, uniqueness and global exponential stability of the system (1.1) will be given in section 4. In section 5, two examples are given to illustrate our theory.

2. Preliminaries

Throughout this paper we assume the following hypotheses:

(H1) There exist constant scalers $F_i > 0$ such that

$$|f_i(x) - f_i(y)| \le F_i|x - y|, \quad i = 1, 2, \dots, n$$

for any $x, y \in R$.

(H2) The delay kernels $K_{ij} : [0, +\infty) \to [0, +\infty)$ are piecewise continuous functions and satisfies:
 (i) $\int_0^\infty K_{ij}(s)ds = 1, \quad i, j = 1, 2, \dots, n.$
 (ii) $\int_0^\infty sK_{ij}(s)ds < \infty, \quad i, j = 1, 2, \dots, n.$
 (iii) There exists a positive number μ such that

$$\int_0^\infty se^{\mu s}K_{ij}(s)ds < \infty, \quad i, j = 1, 2, \dots, n.$$

Let $C = C((-\infty, 0], \mathbb{R}^n)$ be the linear space of bounded and continuous functions which map $(-\infty, 0]$ into $\mathbb{R}^n$. The initial conditions associated with model (1.1) are of the form

$$x_i(t) = \varphi_i(t), \quad -\infty < t \le 0 \tag{2.1}$$

in which $\varphi_i(\cdot)$ are bounded continuous $(i = 1, 2, \dots, n)$.

First, we introduce some notation and recall some basic definitions. For an $n \times n$ matrix, $|A|$ denotes the absolute value matrix given by $|A| = (|a_{ij}|)_{n\times n}$; A^{-1} denotes the inverse of A. Let A, B be two $n \times n$ matrices, $A > B$ represents $a_{ij} > b_{ij}$ for all $i, j = 1, 2, \dots, n$. Let a vector norm $\|x\|_p$ $(p = 1, \infty)$ (simply denoted by $\|x\|$) for $x \in \mathbb{R}^n$ be defined as

$$\|x\|_1 = \sum_{i=1}^n |x_i|, \quad \|x\|_\infty = \max_{1\le i\le n} |x_i|.$$

For $\varphi \in C$, $\|\varphi\|_\infty$ is defined as

$$\|\varphi\|_\infty = \sup_{-\infty<s\le 0} \|\varphi(s)\|_\infty = \sup_{-\infty<s\le 0} \max_{1\le i\le n} |\varphi_i(s)|.$$

Definition 2.1. A function $x : (-\infty, +\infty) \to \mathbb{R}^n$ is said to be the special solution of system (1.1) with initial condition (2.1) if the following two conditions are satisfied

 (i) x is piecewise continuous with first kind discontinuity at the points t_k, $k = 1, 2, \dots$. Moreover, x is left continuous at each discontinuity point.
 (ii) x satisfies model (1.1) for $t \ge 0$, and $x(s) = \varphi(s)$ for $s \in (-\infty, 0]$.

Especially, a point $x^* \in \mathbb{R}^n$ is called an equilibrium point of model (1.1), if $x(t) = x^*$ is a solution of (1.1).

Henceforth, we let $x(t, \varphi)$ denote the special solution of (1.1) with initial condition $\varphi \in C$.

Definition 2.2. The periodic solution $x(t,\varphi)$ of system (1.1) is said to be globally exponentially stable, if there exist positive constants α and M such that every solution $x(t,\phi)$ of (2.1) satisfies

$$\|x(t,\phi)-x(t,\varphi)\|_\infty \le M\|\phi-\varphi\|_\infty e^{-\alpha t}, \quad \forall t\ge 0.$$

Definition 2.3 ([4])**.** A real matrix $D=(d_{ij})_{n\times n}$ is said to be a non-singular M-matrix if $a_{ij}\le 0$, $i,j=1,2,\dots,n$, $i\ne j$, and all successive principal minors of D are positive.

For the non-singular M-matrix, we have the following result.

Lemma 2.4 ([4])**.** *Each of the following conditions is equivalent:*

- (i) *D is a nonsingular M-matrix.*
- (ii) *D^{-1} exists and D^{-1} is a nonnegative matrix.*
- (iii) *The diagonal elements of D are all positive and there exists a positive vector d such that $Dd>0$ or $D^Td>0$.*

Lemma 2.5 ([22])**.** *Suppose y and $\bar{y}$ are two state of model* (1.1)*, then we have*

$$\Big|\bigwedge_{j=1}^{n}\alpha_{ij}f_j(y_j)-\bigwedge_{j=1}^{n}\alpha_{ij}f_j(\bar{y}_j)\Big|\le\sum_{j=1}^{n}\Big|\alpha_{ij}\Big|\cdot\Big|f_j(y_j)-f_j(\bar{y}_j)\Big|,$$
$$\Big|\bigvee_{j=1}^{n}\beta_{ij}f_j(y_j)-\bigvee_{j=1}^{n}\beta_{ij}f_j(\bar{y}_j)\Big|\le\sum_{j=1}^{n}\Big|\beta_{ij}\Big|\cdot\Big|f_j(y_j)-f_j(\bar{y}_j)\Big|.$$

3. Global exponential stability of equilibrium point

In this section, we will give several sufficient conditions on the global exponential stability of equilibrium point for the impulsive FCNN with distributed delays. Consider the case of model (1.1) as $I_i(t)=I_i$, $u_i(t)=u_i$, $i=1,2,\dots,n$, and let $\tilde{I}_i=\sum_{j=1}^{n}\tilde{a}_{ij}u_j+I_i+\bigwedge_{j=1}^{n}T_{ij}u_j+\bigvee_{j=1}^{n}H_{ij}u_j$, then model (1.1) becomes

$$\begin{aligned}\frac{dx_i(t)}{dt}&=-d_ix_i(t)+\sum_{j=1}^{n}a_{ij}f_j(x_j(t))+\bigwedge_{j=1}^{n}b_{ij}\int_{-\infty}^{t}K_{ij}(t-s)f_j(x_j(s))ds\\&\quad+\bigvee_{j=1}^{n}\tilde{b}_{ij}\int_{-\infty}^{t}K_{ij}(t-s)f_j(x_j(s))ds+\tilde{I}_i,\quad t\ne t_k,\\&\Delta x_i(t_k)=x_i(t_k^+)-x_i(t_k^-)=\Delta_k(x_i(t_k)),\quad t=t_k,\end{aligned}\tag{3.1}$$

for $i=1,2,\dots,n$.

Theorem 3.1. *Under assumptions* (H1), (H2)*, the first equation in system* (3.1) *has a unique equilibrium point if $D-(|A|+|B|+|\tilde{B}|)F$ is a nonsingular M-matrix, where $D=\operatorname{diag}(d_1,d_2,\dots,d_n)$, $A=(a_{ij})_{n\times n}$, $B=(b_{ij})_{n\times n}$, $\tilde{B}=(\tilde{b}_{ij})_{n\times n}$, $F=\operatorname{diag}(F_1,F_2,\dots,F_n)$.*

Proof. Let $x^*=(x_1^*,x_2^*,\dots,x_n^*)^T$ denote an equilibrium point of the first equation in model (3.1). Then x^* satisfies

$$d_ix_i^*-\sum_{j=1}^{n}a_{ij}f_j(x_j^*)-\bigwedge_{j=1}^{n}b_{ij}f_j(x_j^*)-\bigvee_{j=1}^{n}\tilde{b}_{ij}f_j(x_j^*)-\tilde{I}_i=0,\quad i=1,2,\dots,n.\tag{3.2}$$

Let

$$h_i(x_i) = d_i x_i - \sum_{j=1}^{n} a_{ij} f_j(x_j) - \bigwedge_{j=1}^{n} b_{ij} f_j(x_j) - \bigvee_{j=1}^{n} \tilde{b}_{ij} f_j(x_j) - \tilde{I}_i = 0, \quad i = 1, 2, \dots, n.$$

Obviously, the solutions of the above system are the equilibrium point of model (3.1). Let us define homotopic mapping

$$H(x, \lambda) = \lambda h(x) + (1 - \lambda)x,$$

where $\lambda \in [0, 1]$, and

$$h(x) = (h_1(x_1), h_2(x_2), \dots, h_n(x_n))^T,$$
$$H(x, \lambda) = (H_1(x_1, \lambda), H_2(x_2, \lambda), \dots, H_n(x_n, \lambda))^T,$$

then for $i \in \{1, 2, \dots, n\}$, from (H1) and Lemma 2.5, we have

$$\begin{aligned}
&|H_i(x, \lambda)| \\
&= \Big|\lambda\Big[d_i x_i - \sum_{j=1}^{n} a_{ij} f_j(x_j) - \bigwedge_{j=1}^{n} b_{ij} f_j(x_j) - \bigvee_{j=1}^{n} \tilde{b}_{ij} f_j(x_j) - \tilde{I}_i\Big] + (1 - \lambda)x_i\Big| \\
&\geq |\lambda d_i x_i + (1 - \lambda)x_i| - \lambda \sum_{j=1}^{n} |a_{ij}||f_j(x_j)| - \lambda \sum_{j=1}^{n} |b_{ij}||f_j(x_j)| \\
&\quad - \lambda \sum_{j=1}^{n} |\tilde{b}_{ij}||f_j(x_j)| - \lambda|\tilde{I}_i| \\
&\geq [1 + \lambda(d_i - 1)]|x_i| - \lambda \sum_{j=1}^{n} |a_{ij}|F_j|x_j| - \lambda \sum_{j=1}^{n} |b_{ij}|F_j|x_j| - \lambda \sum_{j=1}^{n} |\tilde{b}_{ij}|F_j|x_j| \\
&\quad - \lambda\Big[|\tilde{I}_i| + \sum_{j=1}^{n} |a_{ij}||f_j(0)| + \sum_{j=1}^{n} |b_{ij}|f_j(0)| + \sum_{j=1}^{n} |\tilde{b}_{ij}|f_j(0)|\Big]. \\
&= [1 + \lambda(d_i - 1)]|x_i| - \lambda \sum_{j=1}^{n} F_j|x_j|\Big(|a_{ij}| + |b_{ij}| + |\tilde{b}_{ij}|\Big) \\
&\quad - \lambda\Big[|\tilde{I}_i| + \sum_{j=1}^{n} |f_j(0)|\Big(|a_{ij}| + |b_{ij}| + |\tilde{b}_{ij}|\Big)\Big].
\end{aligned}$$

Since $D - (|A| + |B| + |\tilde{B}|)F$ is a nonsingular M-matrix, there exist constants $l_i > 0$ such that

$$l_i d_i - F_i \sum_{j=1}^{n} l_j \Big(|a_{ji}| + |b_{ji}| + |\tilde{b}_{ji}|\Big) > 0, \quad i = 1, 2, \dots, n,$$

then, we have

$$\begin{aligned}
&\sum_{i=1}^{n} l_i |H_i(x, \lambda)| \\
&\geq \sum_{i=1}^{n} l_i(1 - \lambda)|x_i| + \lambda \sum_{i=1}^{n} \Big[d_i l_i |x_i| - l_i \sum_{j=1}^{n} F_j|x_j|\Big(|a_{ij}| + |b_{ij}| + |\tilde{b}_{ij}|\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&\quad - \lambda \sum_{i=1}^{n} l_i \Big[|\tilde{I}_i| + \sum_{j=1}^{n} |f_j(0)| \Big(|a_{ij}| + |b_{ij}| + |\tilde{b}_{ij}| \Big) \Big] \\
&\geq \lambda \sum_{i=1}^{n} \Big[d_i l_i |x_i| - l_i \sum_{j=1}^{n} F_j |x_j| \Big(|a_{ij}| + |b_{ij}| + |\tilde{b}_{ij}| \Big) \Big] \\
&\quad - \lambda \sum_{i=1}^{n} l_i \Big[|\tilde{I}_i| + \sum_{j=1}^{n} |f_j(0)| \Big(|a_{ij}| + |b_{ij}| + |\tilde{b}_{ij}| \Big) \Big] \\
&= \lambda \sum_{i=1}^{n} \Big[l_i d_i - F_i \sum_{j=1}^{n} l_j \Big(|a_{ji}| + |b_{ji}| + |\tilde{b}_{ji}| \Big) \Big] |x_i| \\
&\quad - \lambda \sum_{i=1}^{n} l_i \Big[|\tilde{I}_i| + \sum_{j=1}^{n} |f_j(0)| \Big(|a_{ij}| + |b_{ij}| + |\tilde{b}_{ij}| \Big) \Big] \\
&\geq \lambda l_0 \|x\|_1 - \lambda n I_0 .
\end{aligned}
$$

Define

$$
\begin{gathered}
l_0 = \min_{1 \leq i \leq n} \Big\{ l_i d_i - F_i \sum_{j=1}^{n} l_j \Big(|a_{ji}| + |b_{ji}| + |\tilde{b}_{ji}| \Big) \Big\}, \\
I_0 = \max_{1 \leq i \leq n} \Big\{ l_i \Big(|\tilde{I}_i| + \sum_{j=1}^{n} |f_j(0)| \Big(|a_{ij}| + |b_{ij}| + |\tilde{b}_{ij}| \Big) \Big\},
\end{gathered}
$$

and let

$$
\Gamma = \big\{ x : \|x\|_1 \leq \frac{n(I_0 + 1)}{l_0} \big\}.
$$

Then it follows that $\|x\|_1 = n(I_0 + 1)/l_0$ for any $x \in \partial\Gamma$, and

$$
\sum_{j=1}^{n} l_i |H_i(x, \lambda)| \geq \lambda l_0 \frac{n(I_0 + 1)}{l_0} - \lambda n I_0 > 0, \quad \forall \lambda \in (0, 1],
$$

that is $F(x, \lambda) \neq 0$, for any $x \in \partial\Gamma$, $\lambda \in (0, 1]$. Also, as $\lambda = 0$, $H(x, \lambda) = i_d(x) = x \neq 0$, for any $x \in \partial\Gamma$, here, i_d is identity mapping. Hence, we have $H(x, \lambda) \neq 0$, for any $x \in \partial\Gamma$, $\lambda \in [0, 1]$.

From (H1), it is easy to prove $\deg(i_d, \Gamma, 0) = 1$ thus we have from homotopy invariance theorem [9] that

$$
\deg(h, \Gamma, 0) = deg(i_d, \Gamma, 0) = 1.
$$

By the topological degree theory, we can conclude that (3.1) has at least one solution in Γ. That is, model (3.1) has at least an equilibrium point.

Suppose $y^* = (y_1^*, y_2^*, \dots, y_n^*)^T$ is also an equilibrium point of model (3.1), then we have

$$
\begin{gathered}
d_i x_i^* - \sum_{j=1}^{n} a_{ij} f_j(x_j^*) - \bigwedge_{j=1}^{n} b_{ij} f_j(x_j^*) - \bigvee_{j=1}^{n} \tilde{b}_{ij} f_j(x_j^*) - \tilde{I}_i = 0, \\
d_i y_i^* - \sum_{j=1}^{n} a_{ij} f_j(y_j^*) - \bigwedge_{j=1}^{n} b_{ij} f_j(y_j^*) - \bigvee_{j=1}^{n} \tilde{b}_{ij} f_j(y_j^*) - \tilde{I}_i = 0,
\end{gathered}
$$

this implies

$$\begin{aligned} d_i(x_i^* - y_i^*) &= \sum_{j=1}^n a_{ij}(f_j(x_j^*) - f_j(y_j^*)) + \bigwedge_{j=1}^n b_{ij} f_j(x_j^*) - \bigwedge_{j=1}^n b_{ij} f_j(y_j^*) \\ &\quad + \bigvee_{j=1}^n \tilde{b}_{ij} f_j(x_j^*) - \bigvee_{j=1}^n \tilde{b}_{ij} f_j(y_j^*) \\ &\le \sum_{j=1}^n |a_{ij}||f_j(x_j^*) - f_j(y_j^*)| + |\bigwedge_{j=1}^n b_{ij} f_j(x_j^*) - \bigwedge_{j=1}^n b_{ij} f_j(y_j^*)| \\ &\quad + |\bigvee_{j=1}^n \tilde{b}_{ij} f_j(x_j^*) - \bigvee_{j=1}^n \tilde{b}_{ij} f_j(y_j^*)| \end{aligned}$$

for $i = 1, 2, \dots, n$. By using (H1) and Lemma 2.5, we have

$$d_i|x_i^* - y_i^*| \le \sum_{j=1}^n F_j|x_j^* - y_j^*|\Big(|a_{ij}| + |b_{ij}| + |\tilde{b}_{ij}|\Big), \quad i = 1, 2, \dots, n,$$

which can be rewritten as

$$(D - (|A| + |B| + |\tilde{B}|)F)(|x_1^* - y_1^*|, |x_2^* - y_2^*|, \dots, |x_n^* - y_n^*|)^T \le 0.$$

Since $D - (|A| + |B| + |\tilde{B}|)F$ is a nonsingular M-matrix, $(D - (|A| + |B| + |\tilde{B}|)F)^{-1}$ is a nonnegative matrix. Thus multiplying both sides of the above inequality by $(D - (|A| + |B| + |\tilde{B}|)F)^{-1}$ does not change the inequality direction, it follows that

$$(|x_1^* - y_1^*|, |x_2^* - y_2^*|, \dots, |x_n^* - y_n^*|)^T \le 0.$$

This implies that $x^* = y^*$. Therefore, the system (3.1) has one unique equilibrium point. □

Theorem 3.2. *Assume that* (H1), (H2) *hold, and* $D - (|A| + |B| + |\tilde{B}|)F$ *is a nonsingular* M*-matrix, furthermore, suppose that the impulsive operator* $\Delta_k(x_i(t_k))$ *satisfies*

$$\Delta_k(x_i(t_k)) = -\delta_{ik}(x_i(t_k) - x_i^*), \quad 0 < \delta_{ik} < 2, \quad i = 1, 2, \dots, n, \ k = 1, 2, \dots. \tag{3.3}$$

Then the equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$ *of system* (3.1) *is globally exponentially stable.*

Proof. From (3.3), we have $\Delta_k(x_i^*) = 0$, and by Theorem 3.1, $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$ is only equilibrium point of the system (3.1). Let $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ be an arbitrary solution of the system (3.1). From assumption (H1) and Lemma

2.5, we obtain that

$$
\begin{aligned}
&\frac{d^+|x_i(t)-x_i^*|}{dt}\\
&= \operatorname{sign}(x_i(t)-x_i^*)\frac{d(x_i(t)-x_i^*)}{dt}\\
&\le -d_i|x_i(t)-x_i^*| + \sum_{j=1}^n |a_{ij}||f_j(x_j(t))-f_j(x_j^*)|\\
&\quad + |\bigwedge_{j=1}^n b_{ij}\int_{-\infty}^t K_{ij}(t-s)f_j(x_j(s))ds - \bigwedge_{j=1}^n b_{ij}\int_{-\infty}^t K_{ij}(t-s)f_j(x_j^*)ds|\\
&\quad + |\bigvee_{j=1}^n \tilde{b}_{ij}\int_{-\infty}^t K_{ij}(t-s)f_j(x_j(s))ds - \bigvee_{j=1}^n \tilde{b}_{ij}\int_{-\infty}^t K_{ij}(t-s)f_j(x_j^*)ds|\\
&\le -d_i|x_i(t)-x_i^*| + \sum_{j=1}^n |a_{ij}||f_j(x_j(t))-f_j(x_j^*)|\\
&\quad + |\sum_{j=1}^n |b_{ij}|\int_{-\infty}^t K_{ij}(t-s)|f_j(x_j(s))-f_j(x_j^*)|ds\\
&\quad + |\sum_{j=1}^n |\tilde{b}_{ij}|\int_{-\infty}^t K_{ij}(t-s)|f_j(x_j(s))-f_j(x_j^*)|ds\\
&\le -d_i|x_i(t)-x_i^*| + \sum_{j=1}^n |a_{ij}|F_j|x_j(t)-x_j^*|\\
&\quad + \sum_{j=1}^n \Big(|b_{ij}|+|\tilde{b}_{ij}|\Big)\int_{-\infty}^t K_{ij}(t-s)F_j|x_j(s)-x_j^*|ds\\
&= -d_i|x_i(t)-x_i^*| + \sum_{j=1}^n F_j|a_{ij}||x_j(t)-x_j^*|\\
&\quad + \sum_{j=1}^n F_j\Big(|b_{ij}|+|\tilde{b}_{ij}|\Big)\int_0^{+\infty} K_{ij}(s)|x_j(t-s)-x_j^*|ds
\end{aligned}
\tag{3.4}
$$

for $t > 0$, $i = 1, 2, \dots, n$, $t \neq t_k$, $k = 1, 2, \dots$. Also,

$$
x_i(t_k^+) - x_i^* = -\delta_{ik}(x_i(t_k)-x_i^*) + x_i(t_k) - x_i^* = (1-\delta_{ik})(x_i(t_k)-x_i^*)
$$

for $i = 1, 2, \dots, n$, $k = 1, 2, \dots$. Hence

$$
|x_i(t_k^+) - x_i^*| \le |1-\delta_{ik}||x_i(t_k)-x_i^*| \le |x_i(t_k)-x_i^*| \tag{3.5}
$$

for $i = 1, 2, \dots, n$, $k = 1, 2, \dots$. Since $D - (|A| + |B| + |\tilde{B}|)F$ is a nonsingular M-matrix, there exist constants $l_i > 0$ such that

$$
l_i d_i - F_i \sum_{j=1}^n l_j\Big(|a_{ji}| + |b_{ji}| + |\tilde{b}_{ji}|\Big) > 0, \quad i = 1, 2, \dots, n. \tag{3.6}
$$

Now, for $i = 1, 2, \dots, n$, we define the functions

$$\tilde{h}_i(\alpha_i) = l_i(d_i - \alpha_i) - F_i \sum_{j=1}^{n} l_j \Big[|a_{ji}| + \Big(|b_{ji}| + |\tilde{b}_{ji}|\Big) \int_0^{+\infty} e^{\alpha_i s} K_{ji}(s) ds\Big],$$

where $\alpha_i \in [0, +\infty)$. Obviously, for $i \in \{1, 2, \dots, n\}$, $\tilde{h}_i(\alpha_i)$ are continuous on $[0, +\infty)$, and from (3.6), we know that $\tilde{h}_i(0) > 0$, for $i \in \{1, 2, \dots, n\}$. Also, for $i \in \{1, 2, \dots, n\}$, we have $\tilde{h}_i(\alpha_i) \to -\infty$ as $\alpha_i \to +\infty$. So there exists α_i^* such that $\tilde{h}_i(\alpha_i^*) = 0$, $i \in \{1, 2, \dots, n\}$. Let $\alpha = \min\{\alpha_1, \alpha_2, \dots, \alpha_n\}$, we get

$$\tilde{h}_i(\alpha) = l_i(d_i - \alpha) - F_i \sum_{j=1}^{n} l_j \Big[|a_{ji}| + \Big(|b_{ji}| + |\tilde{b}_{ji}|\Big) \int_0^{+\infty} e^{\alpha_i s} K_{ji}(s) ds\Big] \geq 0 \quad (3.7)$$

for $i = 1, 2, \dots, n$. Let $y_i(t) = e^{\alpha t}|x_i(t) - x_i^*|$, $i = 1, 2, \dots, n$. Then it follows from (3.4) that

$$\begin{aligned} \frac{d^+ y_i(t)}{dt} &= \alpha e^{\alpha t}|x_i(t) - x_i^*| + e^{\alpha t} \frac{d^+|x_i(t) - x_i^*|}{dt} \\ &\leq -(d_i - \alpha) y_i(t) + \sum_{j=1}^{n} |a_{ij}| F_j y_j(t) \\ &\quad + \sum_{j=1}^{n} \Big(|b_{ij}| + |\tilde{b}_{ij}|\Big) F_j \int_0^{+\infty} e^{\alpha s} K_{ij}(s) y_j(t-s) ds \end{aligned} \quad (3.8)$$

for $t > 0$, $i = 1, 2, \dots, n$, $t \neq t_k$, $k = 1, 2, \dots$. Also, from (3.5), we have

$$y_i(t_k^+) = e^{\alpha t_k^+}|x_i(t_k^+) - x_i^*| \leq e^{\alpha t_k}|x_i(t_k) - x_i^*| = y_i(t_k)$$

for $i = 1, 2, \dots, n$, $k = 1, 2, \dots$. Now, we construct the Lyapunov functional

$$V(t) = \sum_{i=1}^{n} l_i \Big[y_i(t) + \sum_{j=1}^{n} F_j \Big(|b_{ij}| + |\tilde{b}_{ij}|\Big) \int_0^{+\infty} e^{\alpha s} K_{ij}(s) \Big(\int_{t-s}^{t} y_j(r) dr\Big) ds\Big]. \quad (3.9)$$

The derivative of $V(t)$ along with the trajectories of model (3.1) is

$$\begin{aligned} &D^+ V(t) \\ &= \sum_{i=1}^{n} l_i \Big[\frac{d^+ y_i(t)}{dt} + \sum_{j=1}^{n} F_j \Big(|b_{ij}| + |\tilde{b}_{ij}|\Big) y_j(t) \int_0^{+\infty} e^{\alpha s} K_{ij}(s) ds \\ &\quad - \sum_{j=1}^{n} F_j \Big(|b_{ij}| + |\tilde{b}_{ij}|\Big) \int_0^{+\infty} e^{\alpha s} K_{ij}(s) y_j(t-s) ds\Big] \\ &\leq \sum_{i=1}^{n} l_i \Big[-(d_i - \alpha) y_i(t) + \sum_{j=1}^{n} |a_{ij}| F_j y_j(t) \\ &\quad + \sum_{j=1}^{n} F_j \Big(|b_{ij}| + |\tilde{b}_{ij}|\Big) \int_0^{+\infty} e^{\alpha s} K_{ij}(s) y_j(t-s) ds \\ &\quad + \sum_{j=1}^{n} F_j \Big(|b_{ij}| + |\tilde{b}_{ij}|\Big) y_j(t) \int_0^{+\infty} e^{\alpha s} K_{ij}(s) ds \end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{j=1}^{n} F_j\Big(|b_{ij}| + |\tilde{b}_{ij}|\Big) \int_0^{+\infty} e^{\alpha s} K_{ij}(s) y_j(t-s) ds\Big] \\
&= - \sum_{i=1}^{n} l_i (d_i - \alpha) y_i(t) + \sum_{i=1}^{n} \sum_{j=1}^{n} l_i F_j |a_{ij}| y_j(t) \\
&\quad + \sum_{i=1}^{n} \sum_{j=1}^{n} l_i F_j \Big(|b_{ij}| + |\tilde{b}_{ij}|\Big) |y_j(t) \int_0^{+\infty} e^{\alpha s} K_{ij}(s) ds \\
&= \sum_{i=1}^{n} \Big\{ - l_i(d_i - \alpha) + F_i \sum_{j=1}^{n} l_j \Big[|a_{ji}| + \Big(|b_{ji}| + |\tilde{b}_{ji}|\Big) \int_0^{+\infty} e^{\alpha s} K_{ji}(s) ds \Big] \Big\} y_i(t) \le 0
\end{aligned}
$$

for $t > 0$, $t \neq t_k$, $k = 1, 2, \dots$. Also,

$$
\begin{aligned}
V(t_k^+) &= \sum_{i=1}^{n} l_i \Big[y_i(t_k^+) + \sum_{j=1}^{n} F_j \Big(|b_{ij}| + |\tilde{b}_{ij}|\Big) \int_0^{+\infty} e^{\alpha s} K_{ij}(s) \Big(\int_{t_k^+ - s}^{t_k^+} y_j(r) dr \Big) ds \Big] \\
&\le \sum_{i=1}^{n} l_i \Big[y_i(t_k) + \sum_{j=1}^{n} F_j \Big(|b_{ij}| + |\tilde{b}_{ij}|\Big) \int_0^{+\infty} e^{\alpha s} K_{ij}(s) \Big(\int_{t_k - s}^{t_k} y_j(r) dr \Big) ds \Big] \\
&= V(t_k), \quad k = 1, 2, \dots .
\end{aligned}
$$

So, we have $V(t) \le V(0)$, for all $t > 0$. From (3.9), we obtain

$$
V(t) \ge \min_{1 \le i \le n} \{l_i\} \sum_{i=1}^{n} y_i(t). \tag{3.10}
$$

Also,

$$
\begin{aligned}
&V(0) \\
&= \sum_{i=1}^{n} l_i \Big[y_i(0) + \sum_{j=1}^{n} F_j \Big(|b_{ij}| + |\tilde{b}_{ij}|\Big) \int_0^{+\infty} e^{\alpha s} K_{ij}(s) \Big(\int_{-s}^{0} y_j(r) dr \Big) ds \Big] \\
&\le \max_{1 \le i \le n} \{l_i\} \sum_{i=1}^{n} \Big[y_i(0) + \sum_{j=1}^{n} F_j \Big(|b_{ij}| + |\tilde{b}_{ij}|\Big) \int_0^{+\infty} e^{\alpha s} K_{ij}(s) \Big(\int_{-s}^{0} y_j(r) dr \Big) ds \Big] \\
&\le \max_{1 \le i \le n} \{l_i\} \sum_{i=1}^{n} \Big[y_i(0) + \sum_{j=1}^{n} F_j \Big(|b_{ij}| + |\tilde{b}_{ij}|\Big) \int_0^{+\infty} s e^{\alpha s} K_{ij}(s) ds \Big(\sup_{-\infty < r \le 0} y_j(r) \Big) \Big].
\end{aligned}
$$

From the above inequality and (3.10), we have

$$
\begin{aligned}
\sum_{i=1}^{n} y_i(t) \le \frac{\max_{1 \le i \le n}\{l_i\}}{\min_{1 \le i \le n}\{l_i\}} \sum_{i=1}^{n} \Big[y_i(0) \\
+ \sum_{j=1}^{n} F_j \Big(|b_{ij}| + |\tilde{b}_{ij}|\Big) \int_0^{+\infty} s e^{\alpha s} K_{ij}(s) ds \Big(\sup_{-\infty < r \le 0} y_j(r) \Big) \Big],
\end{aligned}
$$

for $t > 0$. It follows from the definition of $y_i(t)$ and the above inequality that

$$
\sum_{i=1}^{n} |x_i(t) - x_i^*| \le M e^{-\alpha t} \sup_{-\infty < s \le 0} \sum_{i=1}^{n} |\varphi_i(s) - x_i^*|
$$

for $t > 0$, where

$$M = \frac{\max_{1\le i\le n}\{l_i\}}{\min_{1\le i\le n}\{l_i\}}\Big[1 + \max_{1\le i\le n}\sum_{j=1}^{n} F_j\Big(|b_{ij}| + |\tilde{b}_{ij}|\Big)\int_0^{+\infty} se^{\alpha s}K_{ij}(s)ds\Big].$$

The proof is completed. ☐

As a direct result of Theorems 3.1, 3.2, we have the following result.

Corollary 3.3. *Assume that* (H1), (H2) *hold, then model* (3.1) *has one unique equilibrium point, if any one of the following conditions is true:*

(i) $d_i > F_i\sum_{j=1}^{n}\Big(|a_{ji}| + |b_{ji}| + |\tilde{b}_{ji}|\Big)$, $i = 1, 2, \dots, n$.
(ii) $d_i > \sum_{j=1}^{n} F_j\Big(|a_{ij}| + |b_{ij}| + |\tilde{b}_{ij}|\Big)$, $i = 1, 2, \dots, n$.
(iii) *There exists a positive vector* $l = (l_1, l_2, \dots, l_n)^T > 0$ *such that*

$$l_i d_i > \sum_{j=1}^{n} l_j F_j\Big(|a_{ij}| + |b_{ij}| + |\tilde{b}_{ij}|\Big), \quad i = 1, 2, \dots, n.$$

Furthermore, suppose that the impulsive operator $\Delta_k(x_i(t_k))$ *satisfies*

$$\Delta_k(x_i(t_k)) = -\delta_{ik}(x_i(t_k) - x_i^*), \quad 0 < \delta_{ik} < 2, \quad i = 1, 2, \dots, n,\ k = 1, 2, \dots .$$

Then the equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$ *of the system* (3.1) *is globally exponentially stable.*

Proof. In fact, any one of the conditions (i)-(iii) can assure, $D - (|A| + |B| + |\tilde{B}|)F$ is a nonsingular M-matrix. ☐

4. Periodic oscillatory solution

In the section, we discuss the existence, uniqueness and global exponential stability of the periodic oscillatory solution of model (1.1). Let $I_i : R \to R$ and $u_i : R \to R$ be continuously periodic function with period ω, i.e. $I_i(t + \omega) = I_i(t)$, $u_i(t + \omega) = u_i(t)$ for $i = 1, 2, \dots, n$. Furthermore, we assume that

(H3) There exists a positive integer q such that

$$t_{k+q} = t_k + \omega, \quad \delta_{i(k+q)} = \delta_{ik}, \quad k = 1, 2, \dots,\ i = 1, 2, \dots, n,$$

where δ_{ik} satisfy $\Delta_k(x_i(t_k)) = x_i(t_k^+) - x_i(t_k^-) = -\delta_{ik}x_i(t_k)$, $0 < \delta_{ik} < 2$.

Theorem 4.1. *Under hypothesis* (H1)–(H3)*, there exists exactly one ω-periodic solution of model* (1.1) *and all other solutions of model* (1.1) *converge exponentially to it as $t \to +\infty$, if $D - (|A| + |B| + |\tilde{B}|)F$ is a nonsingular M-matrix.*

Proof. Let $x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \dots, x_n(t, \phi))^T$ and let

$$x(t, \varphi) = (x_1(t, \varphi), x_2(t, \varphi), \dots, x_n(t, \varphi))^T$$

be an arbitrary pair of solutions of (1.1). Since $D-(|A|+|B|+|\tilde{B}|)F$ is a nonsingular M-matrix, (3.6) and (3.7) hold. Let $\tilde{y}(t) = e^{\alpha t}|x_i(t, \phi) - x_i(t, \varphi)|$, $i = 1, 2, \dots, n$, we easily obtain

$$\begin{aligned}\frac{d^+\tilde{y}_i(t)}{dt} &\le -(d_i-\alpha)\tilde{y}_i(t)+\sum_{j=1}^n |a_{ij}|F_j\tilde{y}_j(t)\\ &\quad+\sum_{j=1}^n F_j\Big(|b_{ij}|+|\tilde{b}_{ij}|\Big)\int_0^{+\infty} e^{\alpha s}K_{ij}(s)\tilde{y}_j(t-s)ds,\end{aligned}\tag{4.1}$$

and

$$\begin{aligned}\tilde{y}_i(t_k^+) &= e^{\alpha t_k^+}|x_i(t_k^+,\phi)-x_i(t_k^+,\varphi)|\\ &= e^{\alpha t_k}|x_i(t_k^-,\phi)-\delta_{ik}x_i(t_k,\phi)-x_i(t_k^-,\varphi)+\delta_{ik}x_i(t_k,\varphi)|\\ &= e^{\alpha t_k}|x_i(t_k,\phi)-\delta_{ik}x_i(t_k,\phi)-x_i(t_k,\varphi)+\delta_{ik}x_i(t_k,\varphi)|\\ &= e^{\alpha t_k}|1-\delta_{ik}||x_i(t_k,\phi)-x_i(t_k,\varphi)|\\ &\le e^{\alpha t_k}|x_i(t_k,\phi)-x_i(t_k,\varphi)| = \tilde{y}_i(t_k).\end{aligned}\tag{4.2}$$

Now, we construct the Lyapunov functional

$$V(t)=\sum_{i=1}^n l_i\Big[\tilde{y}_i(t)+\sum_{j=1}^n F_j\Big(|b_{ij}|+|\tilde{b}_{ij}|\Big)\int_0^{+\infty} e^{\alpha s}K_{ij}(s)\Big(\int_{t-s}^t \tilde{y}_j(r)dr\Big)ds\Big].$$

By a minor modification of the proof of Theorem 3.2, we can easily derive

$$\sum_{i=1}^n |x_i(t,\phi)-x_i(t,\varphi)| \le Me^{-\alpha t}\sup_{-\infty<s\le 0}\sum_{i=1}^n |\phi_i(s)-\varphi_i(s)|$$

for $t\ge 0$, where $M\ge 1$ is constant, $\alpha=\min_{1\le i\le n}\{\alpha_i\}$ from (3.8). Therefore, we have

$$\|x(t,\phi)-x(t,\varphi)\|_\infty \le Me^{-\alpha t}\|\phi-\varphi\|_\infty.\tag{4.3}$$

Below, we prove that the system (1.1) has exactly one ω-periodic solution. For each solution $x(t,\phi)$ of (1.1) and each $t\ge 0$, we define a function $x_t(\phi)$ in this fashion:

$$x_t(\phi)(s)=x(t+s,\phi)\quad \text{for } s\in(-\infty,0].$$

From (4.3), we can choose a positive integer N such that $Me^{-\alpha N\omega}\le \frac{1}{6}$.

Now, define a Poincare mapping $C\to C$ by $P(\varphi)=x_\omega(\varphi)$, then $P^N(\varphi)=x_{N\omega}(\varphi)$. Let $t=N\omega$, then

$$\|P^N(\phi)-P^N(\varphi)\|_\infty \le \frac{1}{6}\|\phi-\varphi\|_\infty.$$

This implies that P^N is a contraction mapping, hence there exists one unique fixed point $\varphi^*\in C$ such that $P^N(\varphi^*)=\varphi^*$.

Since $P^N(P(\varphi^*))=P(P^N(\varphi^*))=P(\varphi^*)$, $P(\varphi^*)\in C$ is also a fixed point of P^N, it follows that $P(\varphi^*)=\varphi^*$, that is $x_\omega(\varphi^*)=\varphi^*$.

Let $x(t,\varphi^*)$ be the solution of model (1.1) through $(0,\varphi^*)$, then $x(t+\omega,\varphi^*)$ is also a solution of model (1.1). Obviously

$$x_{t+\omega}(\varphi^*)=x_t(x_\omega(\varphi^*))=x_t(\varphi^*)$$

for all $t\ge 0$. Hence

$$x(t+\omega,\varphi^*)=x(t,\varphi^*).$$

This shows that $x(t,\varphi^*)$ is exactly one ω-periodic solution of model (1.1), and all solutions of model (1.1) converge exponentially to it as $t\to+\infty$. The proof is completed. □

As a direct result of Theorem 4.1, we have following corollary.

Corollary 4.2. *Under hypothesis* (H1)–(H3)*, there exists exactly one ω-periodic solution of model* (1.1) *and all other solutions of model* (1.1) *converge exponentially to it as $t \to +\infty$, if any one of the following conditions is true:*

(i) $d_i > F_i \sum_{j=1}^n \Big(|a_{ji}| + |b_{ji}| + |\tilde{b}_{ji}|\Big)$, $i = 1, 2, \dots, n$.

(ii) $d_i > \sum_{j=1}^n F_j \Big(|a_{ij}| + |b_{ij}| + |\tilde{b}_{ij}|\Big)$, $i = 1, 2, \dots, n$.

(iii) *There exists a positive vector $l = (l_1, l_2, \dots, l_n)^T > 0$ such that*

$$l_i d_i > \sum_{j=1}^n l_j F_j \Big(|a_{ij}| + |b_{ij}| + |\tilde{b}_{ij}|\Big), \quad i = 1, 2, \dots, n.$$

5. Examples

Example 5.1. Consider the model

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= -d_1 x_1(t) + \sum_{j=1}^2 a_{1j} f_j(x_j(t)) + \sum_{j=1}^2 \tilde{a}_{1j} u_j + I_1 \\
&\quad + \bigwedge_{j=1}^2 b_{1j} \int_{-\infty}^t e^{-(t-s)} f_j(x_j(s)) ds + \bigvee_{j=1}^2 \tilde{b}_{1j} \int_{-\infty}^t e^{-(t-s)} f_j(x_j(s)) ds \\
&\quad + \bigwedge_{j=1}^2 T_{1j} u_j + \bigvee_{j=1}^2 H_{1j} u_j, \quad t \ge 0,\ t \ne t_k \\
\Delta x_1(t_k) &= -(1 + \frac{1}{2}\sin(1+k))(x_1(t_k) - \frac{25}{32}), \quad k = 1, 2, \dots, \\
\frac{dx_2(t)}{dt} &= -d_2 x_2(t) + \sum_{j=1}^2 a_{2j} f_j(x_j(t)) + \sum_{j=1}^2 \tilde{a}_{2j} u_j + I_2 \\
&\quad + \bigwedge_{j=1}^2 b_{2j} \int_{-\infty}^t e^{-(t-s)} f_j(x_j(s)) ds + \bigvee_{j=1}^2 \tilde{b}_{2j} \int_{-\infty}^t e^{-(t-s)} f_j(x_j(s)) ds \\
&\quad + \bigwedge_{j=1}^2 T_{2j} u_j + \bigvee_{j=1}^2 H_{2j} u_j, \quad t \ge 0,\ t \ne t_k \\
\Delta x_2(t_k) &= -(1 + \frac{2}{3}\cos(2k))(x_2(t_k) - \frac{1}{2}), \quad k = 1, 2, \dots,
\end{aligned} \tag{5.1}$$

where $0 < t_1 < t_2 < \dots$ is a strictly increasing sequence such that $\lim_{t\to+\infty} t_k = +\infty$; $f_i(x) = \frac{1}{2}(|x+1| + |x-1|)$, $i = 1, 2$;

$$D = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}, \quad A = \begin{pmatrix} \frac{1}{4} & -1 \\ -3 & +1 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix},$$

$$\tilde{A} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad I_1 = I_2 = 2, \quad u_1 = u_2 = 1, \quad T = (T_{ij}) = E, \quad H = (H_{ij}) = E.$$

We can easily check that (H1) and (H2) hold, and for any $x_1, x_2 \in R$, we have

$$|f_1(x_1) - f_2(x_2)| \le |x_1 - x_2|, \quad i = 1, 2,$$

hence $F_1 = F_2 = 1$. It follows that

$$D - (|A| + |B| + |\tilde{B}|)F = \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}$$

is a nonsingular M-matrix. Also, $\alpha_{1k} = 1 + \frac{1}{2}\sin(1+k)$, $\alpha_{2k} = 1 + \frac{2}{3}\cos(2k)$ such that $0 < \alpha_{ik} < 2$, $i = 1, 2$, $k = 1, 2, \dots$. From Corollary 3.3, we know that neural network model (5.1) has one unique equilibrium point, which is globally exponentially stable. Using MATLAB software, we can get the unique equilibrium point $x^* = \big(\frac{25}{32}, \frac{1}{2})^T$.

Example 5.2. Consider the following impulsive neural network model with distributed delays

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= -d_1 x_1(t) + \sum_{j=1}^{2} a_{1j} f_j(x_j(t)) + \sum_{j=1}^{2} \tilde{a}_{1j} u_j + I_1 \\
&\quad + \bigwedge_{j=1}^{2} b_{1j} \int_{-\infty}^{t} e^{-2(t-s)} f_j(x_j(s)) ds + \bigvee_{j=1}^{2} \tilde{b}_{1j} \int_{-\infty}^{t} e^{-2(t-s)} f_j(x_j(s)) ds \\
&\quad + \bigwedge_{j=1}^{2} T_{1j} u_j + \bigvee_{j=1}^{2} H_{1j} u_j, \quad t \geq 0,\ t \neq t_k \\
\Delta x_1(t_k) &= -(1 + \frac{1}{2}\sin(1+k))(x_1(t_k)), \quad t_k = 0.3 + 2(k-1)\pi, \quad k = 1, 2, \dots, \\
\frac{dx_2(t)}{dt} &= -d_2 x_2(t) + \sum_{j=1}^{2} a_{2j} f_j(x_j(t)) + \sum_{j=1}^{2} \tilde{a}_{2j} u_j + I_2 \\
&\quad + \bigwedge_{j=1}^{2} b_{2j} \int_{-\infty}^{t} e^{-2(t-s)} f_j(x_j(s)) ds + \bigvee_{j=1}^{2} \tilde{b}_{2j} \int_{-\infty}^{t} e^{-2(t-s)} f_j(x_j(s)) ds \\
&\quad + \bigwedge_{j=1}^{2} T_{2j} u_j + \bigvee_{j=1}^{2} H_{2j} u_j, \quad t \geq 0,\ t \neq t_k \\
\Delta x_2(t_k) &= -(1 + \frac{2}{3}\cos(2k))(x_2(t_k)), \quad t_k = 0.3 + 2(k-1)\pi,\ k = 1, 2, \dots,
\end{aligned} \tag{5.2}$$

where $0 < t_1 < t_2 < \dots$ is a strictly increasing sequence such that $\lim_{t\to+\infty} t_k = +\infty$; $f_i(x) = \frac{1}{1+e^{-x}}$, $i = 1, 2$;

$$D = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}, \quad A = \begin{pmatrix} \frac{1}{4} & -1 \\ -3 & +1 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix},$$

$$\tilde{B} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

$I_1 = I_2 = 2\sin t$, $u_1 = u_2 = \cos t$, $T = (T_{ij}) = E$, $H = (H_{ij}) = E$. We can easily check that (H1) and (H2) hold, and that for any $x_1, x_2 \in R$, we have

$$|f_1(x_1) - f_2(x_2)| \leq |x_1 - x_2|, \quad i = 1, 2,$$

hence $F_1 = F_2 = 1$. It follows that

$$D - (|A| + |B| + |\tilde{B}|)F = \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}$$

is a nonsingular M-matrix. Also, $\alpha_{1k} = 1 + \frac{1}{2}\sin(1+k)$, $\alpha_{2k} = 1 + \frac{2}{3}\cos(2k)$ such that $0 < \alpha_{ik} < 2$, $i = 1, 2$, $k = 1, 2, \dots$. From Theorem 4.1, we conclude that there exists exactly one 2π-periodic solution of model (5.2), and all other solutions converge exponentially to this solution as $t \to +\infty$.

Conclusions. Stability and periodic oscillatory behavior are important in the applications and theories of neural networks. By employing the theory of topological degree, M-matrix and Lypunov functional, We have obtained some sufficient conditions ensuring the existence, uniqueness and global exponential stability of both the equilibrium point and the periodic solution for a class of impulsive fuzzy cellular neural networks with distributed delays. It is believed that these results are significant and useful for the design and applications of the fuzzy cellular neural networks.

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Kelin Li
Department of Mathematics, Sichuan University of Science & Engineering, Sichuan 643000, China
E-mail address: lkl@suse.edu.cn

Zuoan Li
Department of Mathematics, Sichuan University of Science & Engineering, Sichuan 643000, China
E-mail address: lizuoan@suse.edu.cn

Qiankun Song
Department of Mathematics, Chongqing Jiaotong University, Chongqing, 400074, China
E-mail address: qiankunsong@163.com