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ON A CONVEX COMBINATION OF SOLUTIONS TO ELLIPTIC VARIATIONAL INEQUALITIES

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Abstract. Let u_{g_i} the unique solutions of an elliptic variational inequality with second member g_i $(i = 1, 2)$. We establish necessary and sufficient conditions for the convex combination $tu_{g_1} + (1 - t)u_{g_2}$, to be equal to the unique solution of the same elliptic variational inequality with second member $tg_1 + (1 - t)g_2$. We also give some examples where this property is valid.

1. Introduction

In the linear problems for partial differential equations this property leads to the well known superposition principle which is classically used for example in Fourier series, variational equalities, etc. In these cases the linear combination (and also a convex combination) of two solutions of a linear problem, associated to two data, is also solution of the same problem with linear (convex) combination of the two data.

However, in general, this property is not true for the solutions of nonlinear problems, for example for the variational inequalities. The variational inequality theory is fundamental in order to solve free boundary problems for partial differential equations, e.g. the dam problem [1]; the one-phase Stefan problem [2]; the obstacle problem [3, 4, 5]; the mathematical foundation of the finite element method [6] and its corresponding numerical analysis [7].

The goal of this paper is to give necessary and sufficiently condition to obtain that this property in valid for a convex combination of the solutions of elliptic variational inequalities.

Let V be an Hilbert space, V' its topological dual, K be a closed convex non empty set in V, g_i in V' for $i = 1$ and 2, and a bilinear form $a : V \times V \to \mathbb{R}$, which is

- symmetric: $a(u, v) = a(v, u)$ for all $(v, u) \in V \times V$,
- continuous: there exists $M > 0$ such that $|a(v, u)| \leq M\|v\|_V |u\|_V$ for all $(v, u) \in V \times V$,
- coercive: there exists $m > 0$ such that $|a(v, v)| \geq m\|v\|_V^2$ for all $v \in V$.

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It is known [8, 9, 10] that for each $g_i \in V'$ there exists a unique solution $u_i \in K$, namely

$$a(u_i, v - u_i) \geq \langle g_i, v - u_i \rangle \quad \forall v \in K \quad i = 1, 2\,, \tag{1.1}$$

where $< u\,,\, v >$ denotes the duality brackets between $u \in V'$ and $v \in V$. Then we can consider $g_i \mapsto u_i = u_{g_i}$ as function from V' to V.

We want to establish necessary and sufficiently conditions for the convex combination $u_3(t) = tu_1 + (1-t)u_2$, with $t \in [0,1]$, to be the unique solution of the elliptic variational inequality (1.1) with second member $g_3(t) = tg_1 + (1-t)g_2$, such that

$$u_{tg_1+(1-t)g_2} = tu_{g_1} + (1-t)u_{g_2} \quad \forall t \in [0,1] \tag{1.2}$$

if and only if

$$a(u_{g_1}, u_{g_2} - u_{g_1}) - \langle g_1, u_{g_2} - u_{g_1} \rangle = 0, \tag{1.3}$$
$$a(u_{g_2}, u_{g_1} - u_{g_2}) - \langle g_2, u_{g_1} - u_{g_2} \rangle = 0. \tag{1.4}$$

This means also that if g_1 and g_2 are two points in V' and u_{g_1} and u_{g_2} are the corresponding closest points in the closed convex K then the closest point $u_{tg_1+(1-t)g_2}$ to $tg_1 + (1-t)g_2$ is equal to $tu_{g_1} + (1-t)u_{g_2}$ for all $t \in [0,1]$ if and only if $u_{g_1} - u_{g_2}$ is orthogonal to both $u_{g_1} - g_1$ and $u_{g_2} - g_2$.

This paper is organized as follows. In Section 2 we establish some preliminary results which allow us, in Section 3, to prove our main result that (1.2) is equivalent to (1.3) and (1.4). We also give in Section 4 some examples where this property is valid.

2. Preliminary results

For $t \in [0,1]$ and $v \in K$, we define the function $f : [0,1] \times K \to \mathbb{R}$, by

$$f(t, v) = a(u_3(t), v - u_3(t)) - \langle g_3(t), v - u_3(t) \rangle \tag{2.1}$$

with $u_3(t) = tu_1 + (1-t)u_2$ and $g_3(t) = tg_1 + (1-t)g_2$.

Lemma 2.1. *For all $t \in [0,1]$ and all $v \in K$ there exist A, $B(v)$, and $C(v)$ such that*

$$f(t, v) = At^2 + B(v)t + C(v) \tag{2.2}$$

where, for u_i is the unique solution of (1.1) with given data $g_i \in V'$ $(i = 1, 2)$

$$A = a(u_1 - u_2, u_1 - u_2) - \langle g_1 - g_2, g_1 - g_2 \rangle \geq 0, \tag{2.3}$$
$$B(v) = a(u_1 - u_2, v - 2u_2) - \langle g_1 - g_2, v - u_2 \rangle - \langle g_2, u_1 - u_2 \rangle, \tag{2.4}$$
$$C(v) = a(u_2, v - u_2) - \langle g_2, v - u_2 \rangle \geq 0 \quad \forall v \in K \tag{2.5}$$

Moreover, we have

$$A + B(v) + C(v) \geq 0 \quad \forall v \in K. \tag{2.6}$$

Proof. Since $v = tv + (1-t)v$ for all t in $[0,1]$, we can write f as

$$\begin{aligned} f(t,v) &= a(tu_1 + (1-t)u_2, t(v-u_1) + (1-t)(v-u_2)) \\ &\quad - \langle tg_1 + (1-t)g_2, t(v-u_1) + (1-t)(v-u_2)\rangle \\ &= t^2[a(u_1, v-u_1) + a(u_1, v-u_2) - a(u_2, v-u_1) \\ &\quad + a(u_2, v-u_2) - \langle g_1, v-u_1\rangle + \langle g_1, v-u_2\rangle \\ &\quad + \langle g_2, v-u_1\rangle - \langle g_2, v-u_2\rangle] \\ &\quad + t[(a(u_1, v-u_2) + a(u_2, v-u_1) - 2a(u_2, v-u_2) \\ &\quad - \langle g_1, v-u_2\rangle - \langle g_2, v-u_1\rangle + 2\langle g_2, v-u_2\rangle] \\ &\quad + [a(u_2, v-u_2) - \langle g_2, v-u_2\rangle] \\ &= At^2 + B(v)t + C(v). \end{aligned}$$

So we have

$$\begin{aligned} A &= [a(u_1, v-u_1) - \langle g_1, v-u_1\rangle] - a(u_2, v-u_1) + \langle g_2, v-u_1\rangle \\ &\quad + [a(u_2, v-u_2) - \langle g_2, v-u_2\rangle] - a(u_1, v-u_2) + \langle g_1, v-u_2\rangle \\ &= a(u_1-u_2, v-u_1) - \langle g_1-g_2, v-u_1\rangle + a(u_2-u_1, v-u_2) \\ &\quad - \langle g_2-g_1, v-u_2\rangle \\ &= a(u_1-u_2, u_2-u_1) - \langle g_1-g_2, u_2-u_1\rangle \end{aligned}$$

and we remark that

$$A = [a(u_1, u_2-u_1) - \langle g_1, u_2-u_1\rangle] + [a(u_2, u_1-u_2) - \langle g_2, u_1-u_2\rangle]$$

as u_i $(i=1,2)$ is solution of (1.1) with g_i then $A \geq 0$ and does not depend on v. So (2.3) holds.

$$\begin{aligned} B(v) &= a(u_1, v-u_2) + a(u_2, v-u_1) - 2a(u_2, v-u_2) \\ &\quad - \langle g_1, v-u_2\rangle - \langle g_2, v-u_1\rangle + 2\langle g_2, v-u_2\rangle \\ &= a(u_1-u_2, v-u_2) + a(u_2, (v-u_1) - (v-u_2)) \\ &\quad - \langle g_1-g_2, v-u_2\rangle - \langle g_2, (v-u_1) - (v-u_2)\rangle \\ &= a(u_1-u_2, v-u_2) + a(u_2, u_2-u_1) \\ &\quad - \langle g_1-g_2, v-u_2\rangle - \langle g_2, u_2-u_1\rangle \\ &= a(u_1-u_2, v-2u_2) - \langle g_1-g_2, v-u_2\rangle - \langle g_2, u_2-u_1\rangle \end{aligned}$$

So (2.4) holds. Also

$$C(v) = a(u_2, v-u_2) - \langle g_2, v-u_2\rangle \geq 0,$$

so as u_2 is the solution of the variational inequality (1.1) with second member g_2, then we have (2.5). Moreover

$$A + B(v) + C(v) = f(1,v) = a(u_1, v-u_1) - \langle g_1, v-u_1\rangle \geq 0 \quad \forall v \in K,$$

then (2.6) holds. □

Now we define

$$\alpha = a(u_1, u_2-u_1) - \langle g_1, u_2-u_1\rangle, \tag{2.7}$$

$$\beta = a(u_2, u_1-u_2) - \langle g_2, u_1-u_2\rangle. \tag{2.8}$$

Lemma 2.2. *Let α and β be defined by (2.7) and (2.8) respectively. Then $\alpha \geq 0$, $\beta \geq 0$, and for all $\lambda \in [0,1]$ we have*

$$C(\lambda u_1 + (1-\lambda)u_2) = \lambda\beta \geq 0, \tag{2.9}$$

$$A = \alpha + \beta \geq 0, \tag{2.10}$$

$$B(\lambda u_1 + (1-\lambda)u_2) = -\lambda\alpha - (1+\lambda)\beta \leq 0. \tag{2.11}$$

Proof. As u_i is a solution of (1.1) with g_i $(i = 1, 2)$ then $\alpha \geq 0$ and $\beta \geq 0$. Taking, in (2.5), $v = \lambda u_1 + (1-\lambda)u_2$ with λ in $[0,1]$, we obtain

$$\begin{aligned} C(v) &= a(u_2, \lambda u_1 + (1-\lambda)u_2 - u_2) - \langle g_2, \lambda u_1 + (1-\lambda)u_2 - u_2\rangle \\ &= \lambda[a(u_2, u_1 - u_2) - \langle g_2, u_1 - u_2\rangle] = \lambda\beta \geq 0. \end{aligned}$$

From (2.3) we deduce that $A = \alpha + \beta \geq 0$. Taking, in (2.4) $v = \lambda u_1 + (1-\lambda)u_2$ with λ in $[0,1]$, we have

$$\begin{aligned} B(v) &= a(u_1, \lambda u_1 + (1-\lambda)u_2 - u_2) + a(u_2, \lambda u_1 + (1-\lambda)u_2 - u_1) \\ &\quad - 2a(u_2, \lambda u_1 + (1-\lambda)u_2 - u_2) - \langle g_1, \lambda u_1 + (1-\lambda)u_2 - u_2\rangle \\ &\quad - \langle g_2, \lambda u_1 + (1-\lambda)u_2 - u_1\rangle + 2\langle g_2, \lambda u_1 + (1-\lambda)u_2 - u_2\rangle \\ &= -\lambda[a(u_1, u_2 - u_1) - \langle g_1, u_2 - u_1\rangle] \\ &\quad - (1+\lambda)[a(u_2, u_1 - u_2) - \langle g_2, u_1 - u_2\rangle] \\ &= -\lambda\alpha - (1+\lambda)\beta \leq 0. \end{aligned}$$

□

3. Main result

In this section we give a positive answer to our question when the equality (1.2) is valid.

Theorem 3.1. *We have*

$$u_{tg_1+(1-t)g_2} = tu_{g_1} + (1-t)u_{g_2} \quad \forall t \in [0,1] \tag{3.1}$$

if and only if

$$\alpha = a(u_{g_1}, u_{g_2} - u_{g_1}) - \langle g_1, u_{g_2} - u_{g_1}\rangle = 0 \tag{3.2}$$

and

$$\beta = a(u_{g_2}, u_{g_1} - u_{g_2}) - \langle g_2, u_{g_1} - u_{g_2}\rangle = 0, \tag{3.3}$$

where $u_4(t) = u_{tg_1+(1-t)g_2}$ is the unique solution of the inequality (1.1) with the second member $g_3(t) = tg_1 + (1-t)g_2$ and $u_3(t) = tu_{g_1} + (1-t)u_{g_2}$ is the convex combination for $t \in [0,1]$, of u_{g_1} and u_{g_2} which are solutions of the variational inequality (1.1), respectively with second members g_1 and g_2.

Proof. Suppose that $\alpha = \beta = 0$. Therefore, $A = \alpha + \beta = 0$, so

$$f(t, v) = B(v)t + C(v),$$

and from (2.5) and (2.6) we have

$$f(0, v) = C(v) \geq 0 \quad \text{and} \quad f(1, v) = B(v) + C(v) \geq 0 \quad \forall v \in K,$$

then we deduce that $f(t, v) \geq 0$ for all $(t, v) \in [0,1] \times K$. Therefore $u_3(t) = tu_{g_1} + (1-t)u_{g_2}$ is the unique solution of (1.1) with the second member $g_3(t) = tg_1 + (1-t)g_2$, so we get (3.1), by the uniqueness of the variational inequality (1.1) for a given $g_3(t)$.

Suppose now that $u_3(t) = u_4(t), \quad \forall t \in [0,1]$. Then from (2.1) and (2.2) we have

$$f(t,v) = At^2 + B(v)t + C(v) \geq 0, \quad \forall t \in [0,1], \quad \forall v \in K.$$

Taking $v = u_1 = u_{g_1}$ (i.e. $\lambda = 1$ in Lemma 2.2), we obtain

$$A = \alpha + \beta \geq 0, \quad B(u_1) = -\alpha - 2\beta \leq 0, \quad C(u_1) = \beta \geq 0\,.$$

Thus the discriminant $\Delta(u_1) = B(u_1)^2 - 4AC(u_1)$ of the quadratic function $t \mapsto f(t,v)$ is equal to α^2. Then there exist two roots $t_1 \geq 0$, $t_2 \geq 0$ with

$$\min(t_1,t_2) \geq 1 \quad \text{or} \quad \max(t_1,t_2) \leq 0.$$

Then the two roots $t_1 = 1$ and $t_2 = \frac{\beta}{\alpha+\beta}$ must not be in $]0,1[$ so t_2 must be equal to 1, which gives $\alpha = 0$.

Taking now $v = u_2 = u_{g_2}$ (i.e. $\lambda = 0$ in Lemma 2.2) we obtain

$$A = \alpha + \beta \geq 0, \quad B(u_2) = -\beta \leq 0, \quad C(u_2) = 0$$

thus the corresponding discriminant is $\Delta(u_2) = B(u_1)^2 - 4AC(u_1) = \beta^2$. Then there exist two roots $t_1 \geq 0$, $t_2 \geq 0$ with

$$\min(t_1,t_2) \geq 1 \quad \text{or} \quad \max(t_1,t_2) \leq 0.$$

Then the two roots $t_1 = 0$ and $t_2 = \frac{\beta}{\alpha+\beta}$ must not be in $]0,1[$ so $t_2 = \frac{\beta}{\alpha+\beta}$ must be equal to 0, which give $\beta = 0$. □

Corollary 3.2. *For $K = V$ the variational inequality* (1.1) *becomes the following variational equality*

$$u \in V : \quad a(u,v) = \langle g, v\rangle \quad \forall v \in V$$

thus $\alpha = \beta = 0$ so

$$u_{tg_1+(1-t)g_2} = tu_{g_1} + (1-t)u_{g_2} \quad \forall t \in [0,1]. \tag{3.4}$$

Remark 3.3. Property (3.4) has been used in [11] to prove the strict convexity of the cost functional for optimal control problems.

4. Applications

Let Ω be an open set in $\mathbb{R}^n$, $V = L^2(\Omega)$, so $V' = V$ and the duality brackets $< \cdot, \cdot >$ becomes the scalar product in V denoted by $(\cdot\,,\cdot)_V$. We use the usual notation $G^+ = \max(G,0)$ and $G^- = (-G)^+$, and

$$G \perp F \Longleftrightarrow (G,F)_V = 0.$$

We have a preliminary result.

Lemma 4.1. *Let $G_i \in L^2(\Omega)$ for $i = 1,2$, then we have*

$$G_1^- \perp G_2^+ \quad \text{and} \quad G_2^- \perp G_1^+ \Longleftrightarrow G_1(x)G_2(x) \geq 0 \quad \text{a.e. in } \Omega$$

Proof. Suppose that $G_1(x)G_2(x) \geq 0$ a.e. in Ω.

If $G_1(x) > 0$ which means $G_1^-(x) = 0$ and $G_1^+(x) = G_1(x)$, then $G_2(x) \geq 0$ implies

$$G_2^-(x) = 0 \quad \text{and} \quad G_2^+(x) = G_2(x);$$

thus

$$G_1^-(x)G_2^+(x) = 0 \quad \text{and} \quad G_2^-(x)G_1^+(x) = 0.$$

If $G_1(x) < 0$ which means $G_1^+(x) = 0$ and $G_1^-(x) = G_1(x)$, then $G_2(x) \leq 0$ implies

$$G_2^+(x) = 0 \quad \text{and} \quad G_2^-(x) = G_2(x);$$

thus we have also

$$G_1^-(x)G_2^+(x) = 0 \quad \text{and} \quad G_2^-(x)G_1^+(x) = 0.$$

So we have

$$\begin{aligned}(G_1^-, G_2^+)_V &= \int_\Omega G_1^-(x)G_2^+(x)dx \\ &= \int_{\Omega\cap\{G_1>0\}} G_1^-(x)G_2^+(x)dx + \int_{\Omega\cap\{G_1<0\}} G_1^-(x)G_2^+(x)dx \\ &\quad + \int_{\Omega\cap\{G_1=0\}} G_1^-(x)G_2^+(x)dx = 0 \quad \Rightarrow G_1^- \perp G_2^+,\end{aligned}$$

and

$$\begin{aligned}(G_2^-, G_1^+)_V &= \int_\Omega G_2^-(x)G_1^+(x)dx \\ &= \int_{\Omega\cap\{G_1>0\}} G_2^-(x)G_1^+(x)dx + \int_{\Omega\cap\{G_1<0\}} G_2^-(x)G_1^+(x)dx \\ &\quad + \int_{\Omega\cap\{G_1=0\}} G_2^-(x)G_1^+(x)dx = 0 \quad \Rightarrow G_2^- \perp G_1^+.\end{aligned}$$

Conversely we have

$$\begin{aligned}0 = (G_1^-, G_2^+)_V = \int_\Omega G_1^-(x)G_2^+(x)dx = \int_{\Omega\cap\{G_1<0\}} (-G_1)(x)G_2^+(x)dx \\ = \int_{\Omega\cap\{G_1<0\}\cap\{G_2>0\}} (-G_1)(x)G_2(x)dx\end{aligned}$$

then

$$(G_1^-, G_2^+)_V = 0 \Longrightarrow \begin{cases} |\{G_1<0\}\cap\{G_2>0\}| = 0, \\ G_1<0 \Rightarrow G_2^+ = 0 \Rightarrow G_2 \le 0 \Rightarrow G_1G_2 \ge 0 \\ G_2>0 \Rightarrow G_1^- = 0 \Rightarrow G_1 \ge 0 \Rightarrow G_1G_2 \ge 0 \end{cases}$$

where $|\omega|$ is the measure of the set ω. We have also

$$\begin{aligned}0 = (G_2^-, G_1^+)_V = \int_\Omega G_2^-(x)G_1^+(x)dx = \int_{\Omega\cap\{G_1>0\}} G_1(x)G_2^-(x)dx \\ = \int_{\Omega\cap\{G_1>0\}\cap\{G_2<0\}} G_1(x)(-G_2)(x)dx\end{aligned}$$

then

$$(G_2^-, G_1^+)_V = 0 \Longrightarrow \begin{cases} |\{G_2<0\}\cap\{G_1>0\}| = 0, \\ G_2<0 \Rightarrow G_1^+ = 0 \Rightarrow G_1 \le 0 \Rightarrow G_2G_1 \ge 0 \\ G_1>0 \Rightarrow G_2^- = 0 \Rightarrow G_2 \ge 0 \Rightarrow G_1G_2 \ge 0. \end{cases}$$

This completes the proof. □

Example 4.2. Let $\psi \in L^2(\Omega)$, $V = L^2(\Omega)$, and

$$K = \{v \in L^2(\Omega) : v \ge \psi\}, \quad a(u\,,\,v) = (u\,,\,v)_V.$$

We have here easily the existence and uniqueness of $u \in K$ such that

$$[a(u, v-u) \ge (g\,,\,v-u)_V \quad \forall v \in K] \Leftrightarrow [(u-g\,,\,v-u)_V \ge 0 \quad \forall v \in K]$$

which is also equivalent to

$$u = P_K(g) = \max(g, \psi) = g + (\psi - g)^+ = \psi + (g - \psi)^+. \tag{4.1}$$

Proof. From (4.1), following [10], we have

$$u - g = (\psi - g)^+ \quad \text{and} \quad v - u = v - \psi - (g - \psi)^+$$

so

$$\begin{aligned}(u - g, v - u)_V &= ((\psi - g)^+, (v - \psi) - (g - \psi)^+)_V \\ &= ((\psi - g)^+, v - \psi)_V - ((\psi - g)^+, (g - \psi)^+)_V\end{aligned}$$

as

$$((\psi - g)^+, (g - \psi)^+)_V = ((g - \psi)^-, (g - \psi)^+)_V = 0$$

then

$$(u - g, v - u)_V = ((\psi - g)^+, v - \psi)_V = \int_\Omega (\psi - g)^+ (v - \psi) dx \geq 0, \quad \forall v \in K.$$

□

Theorem 4.3. *Let $V = L^2(\Omega)$, $\psi \in V$, $K = \{v \in V : v \geq \psi\}$, $a(u, v) = (u, v)_V$. For a given $g_i \in V$, $i = 1, 2$, we associate*

$$u_i = u_{g_i} = g_i + (\psi - g_i)^+ = \max(\psi, g_i).$$

Then we have

$$u_{tg_1+(1-t)g_2} = tu_{g_1} + (1-t)u_{g_2} \ \forall t \in [0, 1] \Leftrightarrow (g_1 - \psi)(g_2 - \psi) \geq 0 \quad \text{a.e. in } \Omega. \tag{4.2}$$

Proof. ¿From the definition of α we have

$$\alpha = a(u_1, u_2 - u_1) - (g_1, u_2 - u_1)_V = (u_1 - g_1, u_2 - u_1)_V.$$

From

$$u_1 - g_1 = (\psi - g_1)^+, \quad u_2 - u_1 = (g_2 - \psi)^+ - (g_1 - \psi)^+,$$

we have

$$\begin{aligned}\alpha &= ((\psi - g_1)^+, (g_2 - \psi)^+ - (g_1 - \psi)^+)_V \\ &= ((\psi - g_1)^+, (g_2 - \psi)^+)_V - ((\psi - g_1)^+, (g_1 - \psi)^+)_V \\ &= \int_\Omega (\psi - g_1)^+(x)(g_2 - \psi)^+(x) dx \\ &= \int_\Omega (g_1 - \psi)^-(x)(g_2 - \psi)^+(x) dx,\end{aligned}$$

then we deduce that

$$\alpha = 0 \Leftrightarrow \int_\Omega (g_1 - \psi)^-(x)(g_2 - \psi)^+(x) dx = 0 \Leftrightarrow (g_1 - \psi)^- \perp (g_2 - \psi)^+.$$

We have also from the definition of β that

$$\beta = a(u_2, u_1 - u_2) - (g_2, u_1 - u_2)_V = (u_2 - g_2, u_1 - u_2)_V$$

and from

$$u_2 - g_2 = (\psi - g_2)^+ \quad u_1 - u_2 = (g_1 - \psi)^+ - (g_2 - \psi)^+$$

we have

$$\begin{aligned}\beta &= ((\psi - g_2)^+, (g_1 - \psi)^+ - (g_2 - \psi)^+)_V \\ &= ((\psi - g_2)^+, (g_1 - \psi)^+)_V - ((\psi - g_2)^+, (g_2 - \psi)^+)_V \\ &= \int_\Omega (\psi - g_2)^+(x)(g_1 - \psi)^+(x)dx \\ &= \int_\Omega (g_2 - \psi)^-(x) \cdot (g_1 - \psi)^+(x)dx,\end{aligned}$$

then we deduce that

$$\beta = 0 \Leftrightarrow \int_\Omega (g_2 - \psi)^-(x)(g_1 - \psi)^+(x)dx = 0 \Leftrightarrow (g_2 - \psi)^- \perp (g_1 - \psi)^+.$$

Using now Lemma 4.1, with $G_i = g_i - \psi$, we deduce that

$$\left.\begin{array}{c}(g_2 - \psi)^- \perp (g_1 - \psi)^+ \\ \text{and} \\ (g_1 - \psi)^- \perp (g_2 - \psi)^+\end{array}\right\} \Leftrightarrow (g_1 - \psi)(g_2 - \psi) \geq 0 \quad \text{a.e. in } \Omega.$$

Moreover

$$(g_2 - \psi)^- \perp (g_1 - \psi)^+ \quad \text{and} \quad (g_1 - \psi)^- \perp (g_2 - \psi)^+ \Leftrightarrow \alpha = \beta = 0,$$

and with Theorem 3.1 we have

$$\alpha = \beta = 0 \Leftrightarrow u_{tg_1 + (1-t)g_2} = tu_{g_1} + (1-t)u_{g_2} \quad \forall t \in [0, 1],$$

which gives us the equivalence (4.2) and completes the proof. □

Example 4.4. Let us consider the following free boundary problem of obstacle type [10],

$$\begin{gathered}V = H^1(\Omega), \quad V_0 = H_0^1(\Omega), \quad H = L^2(\Omega), \quad K = \{v \in V_0 : v \geq 0\}, \\ L(v) = (g\,,\,v)_H, \quad \text{and} \quad a(u\,,\,v) = \int_\Omega \nabla u \nabla v dx, \\ u \in K: \quad a(u\,,\,v - u) \geq (g\,,\,v - u)_H \quad \forall v \in K.\end{gathered} \tag{4.3}$$

We recall here some usual notation. a is a bilinear symmetric, coercive and continuous form, there exist m, M, such that

$$m\|v\|_V^2 \leq |||v|||_V^2 = a(v\,,\,v) \leq M\|v\|_V^2$$

and $((\cdot\,,\,\cdot)) : V \times V \to \mathbb{R}$ such that

$$((u, v)) = a(u, v) \quad \forall (u, v) \in V \times V$$

is the inner scalar product in V. L is linear and continuous form on a Hilbert space V, and also on V_0. Then by Riesz theorem there exists a unique $g^\star \in V$ such that

$$(g\,,\,v)_H = ((g^\star, v))_V \quad \forall v \in V.$$

then (4.3) is equivalent to

$$u \in K: \quad ((u, v - u))_V \geq ((g^\star, v - u))_V \quad \forall v \in K. \tag{4.4}$$

Therefore, $u = \mathcal{P}_K(g^\star)$ is the projection of $g^\star$ on K with the norm $|||\cdot|||_V$. Remark that $V_0 \hookrightarrow H \hookrightarrow V'$ so this exemple is a particular cas of (1.1).

Lemma 4.5. *With the above notation, we have*

$$g \geq 0 \quad in \quad \Omega \Longrightarrow g^\star \geq 0 \quad in \quad \Omega \Longrightarrow u = \mathcal{P}_K(g^\star) = g^{\star+} = g^\star.$$

Proof. As $g^\star = g^{\star+} - g^{\star-}$, then

$$\begin{aligned}((g^{\star+} - g^\star, v - g^{\star+}))_V &= ((g^{\star-}, v - g^{\star+}))_V = ((g^{\star-}, v))_V - ((g^{\star-}, g^{\star+}))_V \\ &= a((g^{\star-}, v)_V - a((g^{\star-}, g^{\star+})_V = \int_\Omega \nabla g^{\star-} \nabla v dx.\end{aligned}$$

From

$$\int_\Omega \nabla g^\star \nabla v dx = ((g^\star, v))_V = (g\,,\,v)_H \quad \forall v \in V_0 \quad \text{and} \quad g \in H(\Omega),$$

by Green formula we have the representation of $g^\star$ given by

$$-\Delta g^\star = g \quad \text{in } \Omega \quad \text{and} \quad g^\star_{|\partial\Omega} = 0. \tag{4.5}$$

Since $g \geq 0$ in Ω, by the maximum principle we have $g^\star \geq 0$ thus $g^{\star-} \equiv 0$, so

$$((g^{\star+} - g^\star, v - g^{\star+}))_V = \int_\Omega \nabla g^{\star-} \nabla v dx = 0$$

which gives $u = g^{\star+}$. □

Theorem 4.6. *Let $V = H^1(\Omega)$, $H = L^2(\Omega)$, $K = \{v \in V : v \geq 0\}$, $a(u,v) = (\nabla u, \nabla v)_{H^n}$. For a given $g_i \geq 0$ in H, $i = 1,2$, we have $u_i = u_{g_i} = g_i^\star$ and*

$$u_{tg_1+(1-t)g_2} = tu_{g_1} + (1-t)u_{g_2} \quad \forall t \in [0,1]. \tag{4.6}$$

Proof. As $g_i \geq 0$ in H, then $u_i = u_{g_i} = g_i^\star$, for $i = 1,2$. Moreover

$$\begin{aligned}\alpha &= a(u_1, u_2 - u_1) - (g_1, u_2 - u_1)_H = a(g_1^\star, g_2^\star - g_1^\star) - (g_1, g_2^\star - g_1^\star)_H \\ &= ((g_1^\star, g_2^\star - g_1^\star))_V - (g_1, g_2^\star - g_1^\star)_H\end{aligned}$$

and by the Riesz theorem

$$\alpha = ((g_1^\star, g_2^\star - g_1^\star))_V - (g_1^\star, g_2^\star - g_1^\star)_V = 0.$$

We have

$$\begin{aligned}\beta &= a(u_2, u_1 - u_2) - (g_2, u_1 - u_2)_H = a(g_2^\star, g_1^\star - g_2^\star) - (g_2, g_1^\star - g_2^\star)_H \\ &= ((g_2^\star, g_1^\star - g_2^\star))_V - (g_2, g_1^\star - g_2^\star)_H\end{aligned}$$

and also by the Riesz theorem

$$\beta = ((g_2^\star, g_1^\star - g_2^\star))_V - (g_2^\star, g_1^\star - g_2^\star)_V = 0.$$

So by Theorem 3.1 we obtain (4.6) and we finish the proof. □

Remark 4.7. In the case $\Omega =]0,1[\subset \mathbb{R}$, (4.5) becomes

$$-g^{\star\prime\prime}(x) = g(x) \quad in \quad]0,1[, \quad g^\star(0) = g^\star(1) = 0.$$

and we can obtain the explicit expression of

$$g^\star(x) = x\int_x^1 g(t)(1-t)dt + (1-x)\int_x^1 tg(t)dt$$

so we have

$$g(t) \geq 0 \quad in \quad [0,1] \Longrightarrow g^\star(x) \geq 0 \quad in \quad [0,1].$$

From Lemma 4.5 and Theorem 4.6 we deduce that $u_g = g^\star$. Moreover if $g(t) = g$ constant then

$$g^\star(x) = \frac{x(1-x)}{2} g.$$

Conclusion. The idea in the Theorem 3.1 is simple, rigourously proved, and is very useful for establishing the strict convexity of cost functionals in optimal control problems from elliptic variational inequalities. To the best of our knowledge this idea is new and can not be found in the literature of elliptic variational inequalities and control theory; see for example [12, 14, 13].

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