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MULTIPLE POSITIVE SOLUTIONS FOR SINGULAR BOUNDARY-VALUE PROBLEMS WITH DERIVATIVE DEPENDENCE ON FINITE AND INFINITE INTERVALS

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Abstract. In this paper, Krasnoselskii's theorem and the fixed point theorem of cone expansion and compression are improved. Using the results obtained, we establish the existence of multiple positive solutions for the singular second-order boundary-value problems with derivative dependance on finite and infinite intervals.

1. Introduction

In [1], by an alternative method to Leray-Schauder and sequential technique, Agarwal and O'Regan considered the singular boundary-value problem

$$\begin{gathered} \frac{1}{p}(py')' + \Phi(t)f(t,y,py') = 0, \quad t \in (0,1) \\ \alpha y(0) - \beta \lim_{t\to 0^+} p(t)y'(t) = 0, \quad y(1) = 0 \end{gathered} \tag{1.1}$$

and obtained the existence of one solution to equation (1.1) when $\alpha = 0$ or $\beta = 0$.

In [23], by a generalization of the Kneser's property (continuum) of the cross-sections of the solutions funnel, Palamides and Galanis considered the following problems

$$\begin{gathered} \frac{1}{p}(py')' + \Phi(t)f(t,y,py') = 0, \quad t \in (0,+\infty) \\ y(0) = 0, \quad \lim_{t\to+\infty} p(t)y'(t) = 0 \end{gathered} \tag{1.2}$$

and also obtained the existence of one positive and monotone unbounded solution.

There are some other results on the existence of at least one solution for equation (1.1), (1.2), and we refer the reader also to [2, 3, 4, 5, 6, 7, 8, 11, 19, 20, 21, 22]. Moreover, under the condition that $p \equiv 1$, $\beta = 0$ and f has no singularity at $x = 0$ and $px' = 0$, in [15], using pairs of lower and upper solutions, Henderson and Thompson considered the existence of three solutions for equation (1.1) and

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in [24], by reducing the equation (1.1) to a quasi-linear one, I.Yermachenko and F.Sadyrbaev obtained the existence of multiple solutions for equation (1.1) also.

Up to now, there are fewer results on the existence of multiple positive solutions to equation (1.1), (1.2) if $f(t,x,px')$ is singular at x and is related to px'. Motivated by this, in this paper, we discuss the existence of multiple positive solutions to equation (1.1), (1.2) when $f(t,x,px')$ is singular at $x=0$.

There are three sections in our paper. In section 2, in order to overcome the difficulty from px', we improve the Krasnoselskii's theorem and fixed point theorem of cone expansion and compression on unbounded set in a Banach space with a special norm. In section 3, we establish special cones, and using obtained theorems, present the existence of multiple positive solutions to equation (1.1). In section 4, we consider the existence of multiple positive solutions to equation (1.2).

2. The improvement of the Krasnoselskii's theorem and fixed point theorem of cone expansion and compression

In this section, we improve the Krasnoselskii's theorem and fixed point theorem of cone expansion and compression in a Banach space with a special norm.

In [12], Granas and Dugudji presented the theory of fixed point index on unbounded open sets which has same basic properties as those in the theory of fixed point index on bounded open sets [14]. The degree theory on bounded open sets and unbounded open sets can be found in [9, 12, 13, 16, 17, 25].

According to the the theory of fixed point index on unbounded open sets in Chapter 4 of [12], it is easy to obtain following result. Let E be a real Banach space containing a cone P.

Lemma 2.1. *Assume $\Omega\subseteq E$, $\theta\in\Omega$, $\Omega\cap P$ is a relatively open set in P. Let $A: P\cap\overline{\Omega}\to P$ be continuous with relatively compact $A(P\cap\overline{\Omega})$. Suppose that*

$$Ax\neq\mu x,\quad \forall x\in P\cap\partial\Omega,\ \mu\geq 1. \tag{2.1}$$

Then $i(A,P\cap\Omega,P)=1$.

Proof. Let $H(t,x)=tAx$, $t\in[0,1]$ and $x\in P\cap\partial\Omega$. Then $H:[0,1]\times(P\cap\overline{\Omega})\to P$ is continuous, and the continuity of $H(t,x)$ in t is uniform with respect to $x\in P\cap\overline{\Omega}$. Moreover, $H(t,P\cap\overline{\Omega})$ is relatively compact for every $t\in[0,1]$. Evidently, $H(t,x)\neq x$ for $x\in P\cap\partial\Omega$ and $0\leq t\leq 1$. Hence, by the homotopy invariance and normality of fixed point index, we have

$$i(A,P\cap\Omega,P)=i(\theta,P\cap\Omega,P)=1.$$

The proof is complete. □

Now we consider a real Banach space in a special case. Assume that E is a linear space and it satisfies three conditions:

1. There is a norm $x\to\|x\|_1$ on $x\in E$ and under $\|\cdot\|_1$, E is a normed linear space (not complete)
2. There is another semi-norm $\|\cdot\|_2$
3. Under $\|x\|=\max\{\|x\|_1,\|x\|_2\}$, E is a Banach space.

For example, for $x\in C^1([0,1],R)$, under $\|x\|_1=\max_{t\in[0,1]}|x(t)|$, $C^1([0,1],R)$ is an incomplete normed linear space. Let $\|x\|_2=\max_{t\in[0,1]}|x'(t)|$. Obviously, $\|\cdot\|_2$ is a semi-norm of $C^1([0,1],R)$. If we define $\|x\|=\max\{\|x\|_1,\|x\|_2\}$, $C^1([0,1],R)$ is a Banach space.

Assume that P is a cone of E and $\Omega \subset E$ is a open set with $\sup_{x\in\overline{\Omega}} \|x\|_1 < +\infty$. Since $\sup_{x\in\overline{\Omega}} \|x\|_1 \le \sup_{x\in\overline{\Omega}} \|x\|$, it is possible that Ω is unbounded in E. We have the following lemma(the ideas coming from [14]).

Lemma 2.2. *With E and P as above, assume that $\Omega \subseteq E$ is an open set with $\sup_{x\in\overline{\Omega}} \|x\|_1 < +\infty$. Let $A : P\cap\overline{\Omega} \to P$ be continuous with relatively compact $A(P\cap\overline{\Omega})$ and $B : P\cap\partial\Omega \to P$ be continuous with relatively compact $B(P\cap\partial\Omega)$. Suppose that*

(a) $\inf_{x\in P\cap\partial\Omega} \|Bx\|_1 > 0$;
(b) $x - Ax \ne tBx$, *for all $x \in P\cap\partial\Omega$, $t \ge 0$.*

Then, we have

$$i(A, P\cap\Omega, P) = 0. \tag{2.2}$$

Proof. Suppose that the E_1 is a Banach space completion of E under norm $\|x\|_1$. By the extension theorem of Dugundji [10], we can extend B to a continuous operator from $P\cap\overline{\Omega}$ into P such that

$$B(P\cap\overline{\Omega}) \subseteq \overline{\text{co}}B(P\cap\partial(\Omega)) \subseteq (\overline{\text{co}}B(P\cap\partial\Omega))_1, \tag{2.3}$$

where $(\overline{\text{co}}B(P\cap\partial\Omega))_1$ is the closure of $B(P\cap\partial\Omega)$ under the norm $\|\cdot\|_1$ and the followings are similar. Let $F = B(P\cap\partial\Omega)$, then $(\overline{\text{co}}B(P\cap\partial\Omega))_1 = (\overline{\text{co}}F)_1 = (\overline{M})_1$, where

$$M = \{y = \sum_{i=1}^{n} \lambda_i y_i : y_i \in F, \lambda_i \ge 0, \sum_{i=1}^{n} \lambda_i = 1; n = 1, 2, \dots\}.$$

We first prove

$$\inf_{y\in(\overline{M})_1} \|y\|_1 > 0. \tag{2.4}$$

Denote by E_0 the subspace of E spanned by F under norm $\|\cdot\|_1$. Since $B(P\cap\partial\Omega)$ is relatively compact in E under norm $\|\cdot\|$, we know that $B(P\cap\partial\Omega)$ is relatively compact in E_1 under norm $\|\cdot\|_1$. Therefore, E_0 is separable. Evidently, $P_0 = P\cap E_0$ is a cone of E_0 and $F \subseteq P_0$. By property of the cone [14, Theorem 1.4.1], there exists $f_0 \in E_0^*$ such that $f_0(y) > 0$ for any $y \in P_0$ with $y \ne \theta$. We claim that

$$\inf_{y\in F} f_0(y) = \sigma > 0. \tag{2.5}$$

In fact, if $\sigma = 0$, then there exists $\{y_k\} \subseteq F$ such that $f_0(y_k) \to 0$. By the relative compactness of F in E, there is a subsequence $\{y_{k_i}\}$ of $\{y_k\}$ such that $y_{k_i} \to y_0 \in P$ and $y_0 \in E_0$. Then $y_0 \in P_0$, and so $f_0(y_{k_i}) \to f_0(y_0) = 0$. Hence, $y_0 = \theta$ and $\|y_{k_i}\|_1 \to 0$, which contradicts hypothesis (a). Thus, (2.5) holds.

For any $y = \sum_{i=1}^{n} \lambda_i y_i \in M$, where $y_i \in F$, $\lambda_i \ge 0$ and $\sum_{i=1}^{n} \lambda_i = 1$, we have

$$f_0(y) = \sum_{i=1}^{n} \lambda_i f_0(y_i) \ge \sum_{i=1}^{n} \lambda_i \sigma = \sigma,$$

and therefore

$$f_0(y) \ge \sigma, \forall y \in (\overline{M})_1. \tag{2.6}$$

Since $(\overline{M})_1 = (\overline{\text{co}}F)_1$ is compact, there exists a $z_0 \in (\overline{M})_1$ such that

$$\inf_{y\in(\overline{M})_1} \|y\|_1 = \|z_0\|_1. \tag{2.7}$$

By (2.6), $f_0(z_0) \geq \sigma$, and this implies that $z_0 \neq \theta$. It follows therefore from (2.7) that (2.4) holds. By (2.3) and (2.4), we get

$$\inf_{x \in P \cap \overline{\Omega}} \|Bx\|_1 = \sigma > 0. \tag{2.8}$$

Now, it is easy to show that (2.2) holds. In fact, if $i(A, P \cap \Omega, P) \neq 0$, then by the hypothesis (b) and the homotopy invariance property of fixed point index, we have

$$i(A + tB, P \cap \Omega, P) = i(A, P \cap \Omega, P) \neq 0, \forall t > 0.$$

In particular, choosing $t_0 > \frac{a+c}{\sigma}$, where $a = \sup_{x \in \overline{\Omega}} \|x\|_1$ and $c = \sup_{x \in P \cap \overline{\Omega}} \|Ax\|_1$, we have

$$i(A + t_0 B, P \cap \Omega, P) \neq 0,$$

and so, by the solution property of fixed point index, there exists an $x_0 \in P \cap \Omega$ such that $Ax_0 + t_0 Bx_0 = x_0$. Hence

$$t_0 = \frac{\|x_0 - Ax_0\|_1}{\|Bx_0\|_1} \leq \frac{a + c}{\sigma},$$

which is a contradiction. The proof is complete. □

Corollary 2.3. *Assume that Ω is an open set with $\sup_{x \in \overline{\Omega}} \|x\|_1 < +\infty$. Let $A : P \cap \overline{\Omega} \to P$ be continuous with relatively compact $A(P \cap \overline{\Omega})$. If there exists $u_0 > \theta$ such that*

$$x - Ax \neq tu_0, \forall x \in P \cap \partial\Omega, t \geq 0, \tag{2.9}$$

then (2.2) holds.

Proof. Since $\|\cdot\|_1$ is a norm, $u_0 > \theta$ implies that $\|u_0\|_1 > 0$. Hence, the corollary follows directly from Lemma 2.2 by putting $Bx = u_0$ for any $x \in P \cap \partial\Omega$. □

Corollary 2.4. *Assume that Ω is an open set with $\sup_{x \in \overline{\Omega}} \|x\|_1 < +\infty$. Let $A : P \cap \overline{\Omega} \to P$ be continuous with relatively compact $A(P \cap \overline{\Omega})$. If*

$$Ax \nleq x, \quad \forall x \in P \cap \partial\Omega, \tag{2.10}$$

then (2.2) holds.

Proof. Choose an $u_0 > \theta$. Then,

$$x - Ax \neq tu_0, \forall x \in P \cap \partial\Omega, \ t \geq 0.$$

By Corollary 2.3, (2.2) holds. □

Lemma 2.5. *Assume that Ω is an open set with $\sup_{x \in \overline{\Omega}} \|x\|_1 < +\infty$. Let $A : P \cap \overline{\Omega} \to P$ be continuous with relatively compact $A(P \cap \overline{\Omega})$. Suppose that*

(i) *$\inf_{x \in P \cap \partial\Omega} \|Ax\|_1 > 0$; and*
(ii) *$Ax \neq \mu x, \forall x \in P \cap \partial\Omega, 0 < \mu < 1$.*

Then (2.2) holds.

Proof. Taking $B = A$ in Lemma 2.2, we see that condition (a) of Lemma 2.2 is the same as condition (i) of Lemma 2.5. Also, condition (b) of Lemma 2.2 is true. In fact, if there exist $x_0 \in P \cap \partial\Omega$ and $t_0 \geq 0$ such that $x_0 - Ax_0 = t_0 Ax_0$, then $Ax_0 = \mu x_0$, where $\mu_0 = (1 + t_0)^{-1}$. Evidently $0 < \mu_0 \leq 1$, which contradicts the condition (ii). Thus, (2.2) follows from Lemma 2.2. □

Lemma 2.6. *Assume that Ω is an open set with $\sup_{x \in \overline{\Omega}} \|x\|_1 < +\infty$. Let $A : P \cap \overline{\Omega} \to P$ be continuous with relatively compact $A(P \cap \overline{\Omega})$. Suppose that*

(i') *$Ax \neq \mu x, \forall x \in P \cap \partial\Omega, 0 \leq \mu \leq 1$, and*
(ii') *the set $\{\|Ax\|_1^{-1}Ax | x \in P \cap \partial\Omega\}$ is relatively compact.*

Then (2.2) *holds.*

Proof. Let $A_1x = \alpha(\|Ax\|_1)^{-1}Ax$ for $x \in P\cap\partial\Omega$, where $\alpha = \sup_{x\in P\cap\partial\Omega} \|Ax\|_1 > 0$. Then, by hypotheses, $A_1 : P \cap \partial\Omega \to P$ is continuous with relatively compact $A_1(P\cap\partial\Omega)$. By the extension theorem, A_1 can be extended to a continuous operator from $P\cap\overline{\Omega}$ into P with relatively compact $A_1(P\cap\Omega)$. We now prove that A_1 satisfies the condition (i) and (ii) of Lemma 2.5. In fact, first we have

$$\inf_{x\in P\cap\partial\Omega} \|A_1x\|_1 = \sigma > 0.$$

Secondly, if there exists $x_0 \in P \cap \partial\Omega$ and $0 < \mu_0 \leq 1$ such that $A_1x_0 = \mu_0x_0$, then $Ax_0 = \lambda_0x_0$, where $\lambda_0 = \mu_0\alpha^{-1}\|Ax_0\|_1$. Evidently, $0 < \lambda_0 \leq \mu_0 \leq 1$, which contradicts hypothesis (i'). Hence, by Lemma 2.5, we have

$$i(A_1, P \cap \Omega, P) = 0. \tag{2.11}$$

Now, we prove

$$(1 - t)Ax + tA_1x \neq x, \forall x \in P \cap \partial\Omega, 0 \leq t \leq 1. \tag{2.12}$$

If there is an $x_1 \in P\cap\partial\Omega$ and a $0 \leq t_1 \leq 1$ such that $(1-t_1)Ax_1+t_1A_1x_1 = x_1$, then $Ax_1 = \mu_1x_1$, where $\mu_1 = [1 + t_1(\alpha/\|Ax_1\|_1 - 1)]^{-1}$, $0 \leq \mu_1 \leq 1$, in contradiction with hypothesis (i'). Hence, by (2.11), (2.12) and the homotopy invariance of fixed point index, we get

$$i(A, P \cap \Omega, P) = i(A_1, P \cap \Omega, P) = 0.$$

The proof is complete. □

Theorem 2.7. *Let Ω_1 and Ω_2 be two open in E such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$ with $\sup_{t\in\overline{\Omega}_2} \|x\|_1 < +\infty$. Let $A : P \cap (\overline{\Omega}_2 - \Omega_1) \to P$ be continuous with relatively compact $A(P \cap (\overline{\Omega}_2 - \Omega_1))$. Suppose that one of the two conditions*

(H1) *$Ax \ngeq x$, $\forall x \in P \cap \partial\Omega_1$ and $Ax \nleq x$, $\forall x \in P \cap \partial\Omega_2$,*
(H2) *$Ax \nleq x$, $\forall x \in P \cap \partial\Omega_1$ and $Ax \ngeq x$, $\forall x \in P \cap \partial\Omega_2$*

is satisfied. Then A has at least one fixed point in $P \cap (\Omega_2 - \overline{\Omega}_1)$.

Proof. By the extension theorem (Dugundji [10]), A has a completely continuous extension (also noted by A) from $P \cap \overline{\Omega}_2$ from $P \cap \overline{\Omega}_2$ to P. First we assume that (H1) is satisfied, i.e., it is the case of cone expansion. It is easy to see that

$$Ax \neq \mu x, \quad \forall x \in P \cap \partial\Omega_1,\ \mu \geq 1, \tag{2.13}$$

since, otherwise, there exists $x_0 \in P \cap \partial\Omega_1$ and $\mu_0 \geq 1$ such that $Ax_0 = \mu_0x_0 \geq x_0$, in contradiction with (H1). Now, from (2.1) and Lemma 2.1, we obtain

$$i(A, P \cap \Omega_1, P) = 1. \tag{2.14}$$

On the other hand, by Corollary 2.4, we have

$$i(A, P \cap \Omega_2, P) = 0. \tag{2.15}$$

It follows therefore from (2.14) and (2.15) and additivity property of fixed point index that

$$i(A, P \cap (\Omega_2 - \overline{\Omega}_1), P) = i(A, P \cap \Omega_2, P) - i(A, P \cap \Omega_1, P) = -1 \neq 0. \tag{2.16}$$

Hence, by the solution property of fixed point index, A has at least one fixed point in $P \cap (\Omega_2 - \overline{\Omega}_1)$.

Similarly, when (H2) is satisfied, instead of (2.14), (2.15), we have $i(A, P \cap \Omega_2, P) = 1$, and $i(A, P \cap (\Omega_2 - \overline{\Omega}_1), P) = 1$. As a result we also can assert that A has at least one fixed point in $P \cap (\Omega_2 - \overline{\Omega}_1)$. The proof is complete. □

We remark that this theorem improves [14, theorem 2.3.3] because the condition that Ω_1 and Ω_2 are bounded is not necessary.

Theorem 2.8. *Let $\Omega_1 = \{x \in E | \|x\|_1 < r\}$ and $\Omega_2 = \{x \in E | \|x\|_1 < R\}$ be two open in E with $r < R$. Let $A : P \cap (\overline{\Omega}_2 - \Omega_1) \to P$ be continuous with relatively compact $A(P \cap (\overline{\Omega}_2 - \Omega_1))$. Suppose that one of the two conditions*

- (H3) *$\|Ax\| \le \|x\|$, for all $x \in P \cap \partial\Omega_1$ and $\|Ax\|_1 \ge \|x\|_1$, for all $x \in P \cap \partial\Omega_2$,*
- (H4) *$\|Ax\|_1 \ge \|x\|_1$, for all $x \in P \cap \partial\Omega_1$ and $\|Ax\| \le \|x\|$, for all $x \in P \cap \partial\Omega_2$*

is satisfied. Then A has at least one fixed point in $P \cap (\Omega_2 - \overline{\Omega}_1)$.

Proof. We only need to prove this theorem under condition (H3), since the proof is similar when (H4) is satisfied. By the extension theorem, A can be extended to a continuous operator from $P \cap \overline{\Omega}_2$ into P with relatively compact $A(P \cap \overline{\Omega}_2)$. We may assume that A has no fixed points on $P \cap \partial\Omega_1$ and $P \cap \partial\Omega_2$. It is easy to see that (2.13) holds, since otherwise, there exist $x_0 \in P \cap \partial\Omega_1$ and $\mu_0 > 1$ such that $Ax_0 = \mu_0 x_0$ and hence $\|Ax_0\| = \mu_0\|x_0\| > \|x_0\|$, in contradiction with (H3). Thus, by (2.13), Lemma 2.1, (2.14) holds.

On the other hand, it is also easy to verify

$$Ax \neq \mu x, \forall x \in P \cap \partial\Omega_2, 0 < \mu < 1. \tag{2.17}$$

In fact, if there are $x_1 \in P \cap \partial\Omega_2$ and $0 < \mu_1 < 1$ such that $Ax_1 = \mu_1 x_1$, then

$$\|Ax_1\|_1 = \mu_1\|x_1\|_1 < \|x_1\|_1,$$

in contradiction with (H3). In addition, by (H3) we have

$$\inf_{x \in P \cap \partial\Omega_2} \|Ax\|_1 \ge \inf_{x \in \partial\Omega_2} \|x\|_1 > 0. \tag{2.18}$$

It follows from (2.17), (2.18) and Lemma 2.5 that (2.15) holds. As before, (2.14) and (2.15) imply (2.16), and therefore A has at least one fixed point in $P \cap (\Omega_2 - \overline{\Omega}_1)$. □

We remark that this theorem improves the the Krasnoselskii's theorem in [14] because the condition that Ω_1 and Ω_2 are bounded is not necessary.

3. The existence of multiple positive solutions to equation (1.1)

In this section, we consider (1.1) and suppose that $f \in C([0,1] \times R_0^+ \times R, R^+)$, $p \in C([0,1], R) \cap C((0,1), R_0^+) \cap C^1((0,1), R)$ with $\int_0^1 \frac{1}{p(r)} dr < +\infty$, $\Phi \in C((0,1), R_0^+)$ and $\alpha \ge 0$, $\beta \ge 0$ (not equal to 0 at the same time); here $R^+ = [0, +\infty)$, $R_0^+ = (0, +\infty)$, $R = (-\infty, +\infty)$. Let

$$\rho^2 = \beta + \alpha \int_0^1 \frac{1}{p(r)} dr \quad (\rho > 0),$$

$$u_1(t) = \frac{1}{\rho} \int_t^1 \frac{1}{p(r)} dr, \quad v_1(t) = \frac{1}{\rho}(\beta + \alpha \int_0^t \frac{1}{p(r)} dr),$$

$$G_1(t,s)=\begin{cases} u_1(t)v_1(s)p(s), & 0\le s\le t\le 1\\ v_1(t)u_1(s)p(s), & 0\le t\le s\le 1.\end{cases}$$

Assume that $C_p^1[0,1]=\{x:[0,1]\to R|\ x(t)$ is continuous on $[0,1]$ and $p(t)x'(t)$ is continuous on $[0,1]$ also with $\max_{t\in[0,1]}p(t)|x'(t)|<+\infty\}$ (see [21]). For $x\in C_p^1$, let $\|x\|_1=\max_{t\in[0,1]}|x(t)|$, $\|x\|_2=\max_{t\in[0,1]}p(t)|x'(t)|$ and $\|x\|=\max\{\|x\|_1,\|x\|_2\}$. It is easy to see that C_p^1 satisfies the conditions (1), (2) and (3) of the Banach space E in section 2.

Obviously, $x(t)\in C_p^1$ is a solution to equation (1.1) if and only if $x(t)$ is a solution of the following integral equation

$$x(t)=\int_0^1 G_1(t,s)\Phi(s)f(s,x(s),p(s)x'(s))ds,\quad t\in[0,1].$$

Let $P=\{x\in C_p^1|x(t)\ge\gamma_1(t)\|x\|_1\}$, where $\gamma_1(t)=u_1(t)v_1(t)\frac{1}{u_1(0)v_1(1)}$ for all $t\in[0,1]$.

Lemma 3.1. *Assume that $l\in L^1[0,1]$ with $l(t)>0$ for all $t\in(0,1)$ and $q(t)=\int_0^1 G_1(t,s)l(s)ds$, $t\in[0,1]$. Then*

$$q(t)\ge\gamma_1(t)\max_{s\in[0,1]}q(s).$$

Proof. Suppose $q(t_0)=\max_{s\in[0,1]}q(s)$. Then

$$\begin{aligned}
\frac{G_1(t,s)}{G_1(t_0,s)}&=\begin{cases}\frac{v_1(t)u_1(s)p(s)}{u_1(t_0)v_1(s)p(s)}, & 0\le t\le s\le t_0\le 1\\ \frac{u_1(t)v_1(s)p(s)}{v_1(t_0)u_1(s)p(s)}, & 0\le t_0\le s\le t\le 1\\ \frac{v_1(t)u_1(s)p(s)}{v_1(t_0)u_1(s)p(s)}, & 0\le t,t_0\le s\le 1\\ \frac{u_1(t)v_1(s)p(s)}{u_1(t_0)v_1(s)p(s)}, & 0\le s\le t,t_0\le 1\end{cases}\\
&=\begin{cases}u_1(t)v_1(t)\frac{u_1(s)}{u_1(t_0)}\frac{1}{v_1(s)u_1(t)}, & 0\le t\le s\le t_0\le 1\\ u_1(t)v_1(t)\frac{v_1(s)}{v_1(t_0)}\frac{1}{u_1(s)v_1(t)}, & 0\le t_0\le s\le t\le 1\\ u_1(t)v_1(t)\frac{1}{u_1(t)v_1(t_0)}, & 0\le t,t_0\le s\le 1\\ u_1(t)v_1(t)\frac{1}{u_1(t_0)v_1(t)}, & 0\le s\le t,t_0\le 1\end{cases}\\
&\ge\begin{cases}u_1(t)v_1(t)\frac{1}{v_1(1)u_1(0)}, & 0\le t\le s\le t_0\le 1\\ u_1(t)v_1(t)\frac{1}{u_1(0)v_1(1)}, & 0\le t_0\le s\le t\le 1\\ u_1(t)v_1(t)\frac{1}{u_1(0)v_1(1)}, & 0\le t,t_0\le s\le 1\\ u_1(t)v_1(t)\frac{1}{u_1(0)v_1(1)}, & 0\le s\le t,t_0\le 1\end{cases}\\
&=\gamma_1(t).
\end{aligned}$$

As a consequence,

$$\begin{aligned}
q(t)&=\int_0^1 G_1(t,s)l(s)ds=\int_0^1\frac{G_1(t,s)}{G_1(t_0,s)}G_1(t_0,s)l(s)ds\\
&\ge\gamma_1(t)\int_0^1 G_1(t_0,s)l(s)ds=\gamma_1(t)\max_{s\in[0,1]}q(s).
\end{aligned}$$

The proof is complete. ∎

Now we will list some conditions for convenience:

(H1) There exists a $k \in C([0,1], R_0^+)$, a $g \in C(R_0^+, R_0^+)$ and a decreasing continuous function $h \in C(R_0^+, R_0^+)$ such that
$$f(t,x,z) \le k(t)g(x), \quad \forall x \in R_0^+, z \in R, t \in [0,1],$$
where $\frac{g(x)}{h(x)}$ is an increasing function and $\int_0^1 p(s)\Phi(s)k(s)h(c\gamma_1(s))ds < +\infty$ for each $c > 0$;

(H2)
$$\sup_{c \in R_0^+} \frac{ch(c)}{u_1(0)v_1(1)\int_0^1 p(s)\Phi(s)k(s)h(c\gamma_1(s))dsg(c)} > 1;$$

(H3) There exists a $k_1 \in C([0,1], R_0^+)$ and a $g_1 \in C(R_0^+, R_0^+)$ with $f(t,x,z) \ge k_1(t)g_1(x)$, for all $(t,x,z) \in [0,1] \times R_0^+ \times (-\infty, +\infty)$ such that
$$\lim_{x \to +\infty} \frac{g_1(x)}{x} = +\infty,$$
where $\int_0^1 p(s)\Phi(s)k_1(s)ds < +\infty$;

(H4) For any $c > 0$, there exists a $\psi_c \in C([0,1], R_0^+)$ such that $f(t,x,z) \ge \psi_c(t)$ for all $(t,x,z) \in [0,1] \times (0,c] \times (-\infty, +\infty)$ with $\int_0^1 p(s)\Phi(s)\psi_c(s)ds < +\infty$.

For given $n \in \{1,2,\dots\}$, let $f_n(t,x,z) = f(t, \max\{\frac{1}{n}, x\}, z)$ and for $x \in P$, define
$$(A_n x)(t) = \int_0^1 G_1(t,s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds, \quad n \in \{1,2,\dots\},\ t \in [0,1]. \tag{3.1}$$

Lemma 3.2. *Assume the condition (H1) holds. Then, for every $n \in \{1,2,\dots\}$, $A_n : P \to P$ is continuous and for any $r > 0$ and $B_r = \{x \in C_p^1 | \|x\|_1 \le r\}$, $A_n(P \cap B_r)$ is relatively compact.*

Proof. First, for a given $n \in \{1,2,\dots\}$, we show that $A_n P \subseteq P$. For any $x \in P$, we have
$$\begin{aligned}
|(A_n x)(t)| &= \Big|\int_0^1 G_1(t,s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds\Big| \\
&\le \int_0^1 G_1(t,s)\Phi(s)f(s,\max\{\frac{1}{n},x(s)\},p(s)x'(s))ds \\
&\le \int_0^1 G_1(t,s)\Phi(s)k(s)g(\max\{\frac{1}{n},x(s)\})ds \\
&\le \int_0^1 G_1(t,s)\Phi(s)k(s)h(\max\{\frac{1}{n},x(s)\})\frac{g(\max\{\frac{1}{n},x(s)\})}{h(\max\{\frac{1}{n},x(s)\})}ds \\
&\le \int_0^1 G_1(t,s)\Phi(s)k(s)dsh(\frac{1}{n})\frac{g(\max\{\frac{1}{n},\|x\|_1\})}{h(\max\{\frac{1}{n},\|x\|_1\})} \\
&< +\infty
\end{aligned}$$
and
$$\begin{aligned}
|p(t)(A_n x)'(t)| = \Big| &-\frac{1}{\rho}\int_0^t v_1(s)p(s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds \\
&+\frac{\alpha}{\rho}\int_t^1 u_1(s)p(s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds\Big|
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{K}{\rho}\int_0^1 p(s)\Phi(s)f(s,\max\{\frac{1}{n},x(s)\},p(s)x'(s))ds\\
&\le \frac{K}{\rho}\int_0^1 p(s)\Phi(s)k(s)g(\max\{\frac{1}{n},x(s)\})ds\\
&\le \frac{K}{\rho}\int_0^1 p(s)\Phi(s)k(s)h(\max\{\frac{1}{n},x(s)\})\frac{g(\max\{\frac{1}{n},x(s)\})}{h(\max\{\frac{1}{n},x(s)\})}ds\\
&\le \frac{K}{\rho}\int_0^1 p(s)\Phi(s)k(s)dsh(\frac{1}{n})\frac{g(\max\{\frac{1}{n},\|x\|_1\})}{h(\max\{\frac{1}{n},\|x\|_1\})}\\
&< +\infty,
\end{aligned}
$$

where $K=\max\{\alpha u_1(0),v_1(1)\}$ and the following is same as before. Then, A_n is well defined. Moreover, from Lemma 3.1, for any $x\in P$, we have

$$
(A_nx)(t)\ge \gamma_1(t)\max_{s\in[0,1]}|(A_nx)(s)|=\gamma_1(t)\|A_nx\|_1,\quad \forall t\in[0,1].
$$

Consequently, $A_nP\subseteq P$.

Second, we show $A_n:P\to P$ is continuous. Assume that $\lim_{m\to+\infty}x_m=x_0$, which means there exists an $M>1/n$ such that $\|x_m\|\le M$, for all $m\in\{0,1,2,\dots\}$. Then

$$
\begin{aligned}
f_n(t,x_m(t),p(t){x'}_m(t)) &\le k(t)g(\max\{\frac{1}{n},x_m(t)\})\\
&= k(t)h(\max\{\frac{1}{n},x_m(t)\})\frac{g(\max\{\frac{1}{n},x_m(t)\})}{h(\max\{\frac{1}{n},x_m(t)\})} \qquad (3.2)\\
&\le k(t)h(\frac{1}{n})\frac{g(M)}{h(M)},\quad m\in\{1,2,\dots\}.
\end{aligned}
$$

Since

$$
f_n(t,x_m(t),p(t)x_m'(t))\to f_n(t,x_0(t),p(t)x_0'(t)),\quad \text{as } m\to+\infty,
$$

from (3.2), the Lesbegue Dominated Convergence Theorem guarantees

$$
\begin{aligned}
&\lim_{m\to+\infty}\max_{t\in[0,1]}|(A_nx_m)(t)-(A_nx_0)(t)|\\
&=\lim_{m\to+\infty}\max_{t\in[0,1]}|\int_0^1 G_1(t,s)\Phi(s)f_n(s,x_m(s),p(s)x_m'(s))ds\\
&\quad -\int_0^1 G_1(t,s)\Phi(s)f_n(s,x_0(s),p(s)x_0'(s))ds|\\
&\le \lim_{m\to+\infty}\max_{t\in[0,1]}\int_0^1 G_1(t,s)\Phi(s)\Big|f_n(s,x_m(s),p(s)x_m'(s))\\
&\quad -f_n(s,x_0(s),p(s)x_0'(s))\Big|ds\\
&\le \lim_{m\to+\infty}u_1(0)v_1(1)\int_0^1 p(s)\Phi(s)\Big|f_n(s,x_m(s),p(s)x_m'(s))\\
&\quad -f_n(s,x_0(s),p(s)x_0'(s))\Big|ds=0
\end{aligned}
$$

and

$$
\lim_{m\to+\infty}\max_{t\in[0,1]}|p(t)(A_nx_m)'(t)-p(t)(A_nx_0)'(t)|
$$

$$
\begin{aligned}
&= \lim_{m\to+\infty}\max_{t\in[0,1]}\Big| -\frac{1}{\rho}\int_0^t v_1(s)p(s)\Phi(s)[f_n(s,x_m(s),p(s)x_m'(s)) \\
&\quad - f_n(s,x_0(s),p(s)x_0'(s))]ds + \frac{\alpha}{\rho}\int_t^1 u_1(s)p(s)\Phi(s)[f_n(s,x_m(s),p(s)x_m'(s)) \\
&\quad - f_n(s,x_0(s),p(s)x_0'(s))]ds\Big| \\
&\le \lim_{m\to+\infty}\frac{K}{\rho}\int_0^1 p(s)\Phi(s)\Big|f_n(s,x_m(s),p(s)x_m'(s)) - f_n(s,x_0(s),p(s)x_0'(s))\Big|ds \\
&= 0,
\end{aligned}
$$

which mean that

$$
\lim_{m\to+\infty}\|A_nx_m - A_nx_0\| = 0.
$$

Finally, we show $A_n(B_r\cap P)$ is relatively compact. Obviously, B_r is an unbounded set in C_p^1. Without loss of generality, we suppose $r > 1/n$. Then, for any $x\in B_r\cap P$, we have

$$
\begin{aligned}
\max_{t\in[0,1]}|(A_nx)(t)| &= \max_{t\in[0,1]}|\int_0^1 G_1(t,s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds| \\
&= \max_{t\in[0,1]}\int_0^1 G_1(t,s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds \\
&\le \max_{t\in[0,1]}\int_0^1 G_1(t,s)\Phi(s)k(s)h(\max\{\frac{1}{n},x(s)\})\frac{g(\max\{\frac{1}{n},x(s)\})}{h(\max\{\frac{1}{n},x(s)\})}ds \\
&\le \max_{t\in[0,1]}\int_0^1 G_1(t,s)\Phi(s)k(s)dsh(\frac{1}{n})\frac{g(r)}{h(r)} \\
&\le u_1(0)v_1(1)\int_0^1 p(s)\Phi(s)k(s)dsh(\frac{1}{n})\frac{g(r)}{h(r)}
\end{aligned}
$$

and

$$
\begin{aligned}
\max_{t\in[0,1]}|p(t)(A_nx)'(t)| &= \max_{t\in[0,1]}|-\frac{1}{\rho}\int_0^t v_1(s)p(s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds \\
&\quad + \frac{\alpha}{\rho}\int_t^1 u_1(s)p(s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds| \\
&\le \frac{K}{\rho}\int_0^1 p(s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds \\
&\le \frac{K}{\rho}\int_0^1 p(s)\Phi(s)k(s)h(\max\{\frac{1}{n},x(s)\})\frac{g(\max\{\frac{1}{n},x(s)\})}{h(\max\{\frac{1}{n},x(s)\})}ds \\
&\le \frac{K}{\rho}\int_0^1 p(s)\Phi(s)k(s)dsh(\frac{1}{n})\frac{g(r)}{h(r)},
\end{aligned}
$$

which means that $A_n(B_r\cap P)$ is bounded. Assume that $t,t'\in[0,1]$. Then, for $x\in B_r\cap P$, we have

$$
\begin{aligned}
|(A_nx)(t) - (A_nx)(t')| &= |\int_0^1 G_1(t,s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds \\
&\quad - \int_0^1 G_1(t',s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds|
\end{aligned}
$$

$$\begin{aligned}
&\le \int_0^1 |G_1(t,s)-G_1(t',s)|\Phi(s)f_n(s,x(s),p(s)x'(s))ds\\
&\le \int_0^1 |G_1(t,s)-G_1(t',s)|\Phi(s)k(s)dsh(\frac{1}{n})\frac{g(r)}{h(r)}
\end{aligned}$$

and

$$\begin{aligned}
&|p(t)(A_nx)'(t)-p(t')(A_nx)'(t')|\\
&=|-\frac{1}{\rho}\int_0^t v_1(s)p(s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds\\
&\quad+\frac{1}{\rho}\int_0^{t'} v_1(s)p(s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds\\
&\quad+\frac{\alpha}{\rho}\int_t^1 u_1(s)p(s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds\\
&\quad-\frac{\alpha}{\rho}\int_{t'}^1 u_1(s)p(s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds|\\
&\le 2\frac{K}{\rho}|\int_t^{t'} p(s)\Phi(s)f_n(s,x(s),p(s)x'(s))|ds\\
&\le 2\frac{K}{\rho}|\int_t^{t'} p(s)\Phi(s)k(s)ds|h(\frac{1}{n})\frac{g(r)}{h(r)}.
\end{aligned}$$

Then, for any $\varepsilon>0$, we can choose $\delta>0$ small enough such that

$$|(A_nx)(t)-(A_nx)(t')|<\varepsilon,\quad |p(t)(A_nx)'(t)-p(t')(A_nx)'(t')|<\varepsilon,$$

for all $x\in B_r\cap P$, $|t-t'|<\delta$, $t,t'\in[0,1]$. Consequently, $\{(A_n(B_r\cap P))(t)\}$ and $\{p(t)(A_n(B_r\cap P))'(t)\}$ is equicontinuous on $[0,1]$.

Consequently, from Arzela-Ascoli theorem, $A_n(B_r\cap P)$ is relatively compact. The proof is complete. □

Theorem 3.3. *Assume that (H1)–(H4) hold. Then* (1.1) *has at least two positive solutions.*

Proof. From Lemma 3.2, for each $n\in\{1,2,\dots\}$, $A_n:P\to P$ is continuous operator and for any $r>0$, $A_n(B_r\cap P)$ is relatively compact. From (H2), choose $R_0>0$ such that

$$\frac{R_0h(R_0)}{u_1(0)v_1(1)\int_0^1 p(s)\Phi(s)k(s)h(R_0\gamma_1(s))dsg(R_0)}>1. \tag{3.3}$$

Without loss of the generality, suppose $R_0\ge 1/n$, $n\in\{1,2,\dots\}$. Set

$$\Omega_1=\{x\in C_p^1|\|x\|_1<R_0\}.$$

Then for any $x\in\partial\Omega_1\cap P$, one has

$$x\not\le A_nx,\quad n\in\{1,2,\dots\}. \tag{3.4}$$

If there exists an $x_0\in\partial\Omega_1\cap P$ such that $x_0\le A_nx_0$, obviously, $\max\{\frac{1}{n},x_0(t)\}\ge x_0(t)\ge\gamma_1(t)R_0$. Then

$$\begin{aligned}
R_0&=\max_{t\in[0,1]}|x_0(t)|\\
&\le \max_{t\in[0,1]}\int_0^1 G_1(t,s)\Phi(s)f(s,\max\{\frac{1}{n},x_0(s)\},p(s)x'_0(s))ds
\end{aligned}$$

$$\begin{aligned}
&\le u_1(0)v_1(1)\int_0^1 p(s)\Phi(s)k(s)g(\max\{\frac{1}{n},x_0(s)\})ds\\
&\le u_1(0)v_1(1)\int_0^1 p(s)\Phi(s)k(s)h(\max\{\frac{1}{n},x_0(s)\})\frac{g(\max\{\frac{1}{n},x_0(s)\})}{h(\max\{\frac{1}{n},x_0(s)\})})ds\\
&\le u_1(0)v_1(1)\int_0^1 p(s)\Phi(s)k(s)h(R_0\gamma_1(s))ds\frac{g(R_0)}{h(R_0)}
\end{aligned}$$

which implies

$$\frac{R_0h(R_0)}{u_1(0)v_1(1)\int_0^1\Phi(s)k(s)h(R_0\gamma_1(s))dsg(R_0)}\le 1.$$

This contradicts (3.3). Then, (3.4) is true. From the proof of Theorem 2.7, the (2.13) is true, which means

$$i(A_n,P\cap\Omega_1,P)=1,\quad n\in\{1,2,\dots\}. \tag{3.5}$$

As a result, for $n\in\{1,2,\dots\}$, there exists an $x_n^{(1)}\in P\cap\Omega_1$ with $A_nx_n^{(1)}=x_n^{(1)}$. Since $\|x_n^{(1)}\|_1\le R_0$, $n\in\{1,2,\dots\}$, it is easy to see that $\{x_n^{(1)}(t)\}$ and $\{p(t){x'}_n^{(1)}(t)\}$ are uniformly bounded. Moreover, from Lemma 3.1, the condition (H4), there exists a $\psi_{R_0}(t)$ such that

$$\begin{aligned}
x_n^{(1)}(t)&=\int_0^1 G_1(t,s)\Phi(s)f(s,\max\{\frac{1}{n},x_n^{(1)}(s)\},p(s)(x_n^{(1)})'(s))ds\\
&\ge\int_0^1 G_1(t,s)\Phi(s)\psi_{R_0}(s)ds\\
&\ge\gamma_1(t)k^*,
\end{aligned} \tag{3.6}$$

where $k^*=\max_{t\in[0,1]}\int_0^1 G_1(t,s)\Phi(s)\psi_{R_0}(s)ds$. Thus, for any t', $t''\in(0,1)$ and $n\in\{1,2,\dots\}$,

$$\begin{aligned}
&|x_n^{(1)}(t')-x_n^{(1)}(t'')|\\
&\le\int_0^1|G_1(t',s)-G_1(t'',s)|f_n(s,x_n^{(1)}(s),p(s)(x_n^{(1)})'(s))ds\\
&\le\int_0^1|G_1(t',s)-G_1(t'',s)|\Phi(s)k(s)h(\gamma_1(s)k^*)ds\frac{g(R_0)}{h(R_0)},
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
&|p(t')(x_n^{(1)})'(t')-p(t'')(x_n^{(1)})'(t'')|\\
&\le 2\Big|\int_{t'}^{t''}p(s)\Phi(s)f_n(s,x_n^{(1)}(s),p(s)(x_n^{(1)})'(s))ds\Big|\\
&\le 2\Big|\int_{t'}^{t''}p(s)\Phi(s)k(s)h(\gamma_1(s)k^*)ds\Big|\frac{g(R_0)}{h(R_0)},\quad n\in\{1,2,\dots\}.
\end{aligned} \tag{3.8}$$

Consequently, for any $\varepsilon>0$, we can choose a $\delta>0$ such that

$$|x_n^{(1)}(t')-x_n^{(1)}(t'')|<\varepsilon,\quad |p(t')(x_n^{(1)})'(t')-p(t'')(x_n^{(1)})'(t'')|<\varepsilon,$$

for all $n\in\{1,2,\dots\}$, $|t'-t''|<\delta$, $t',t''\in[0,1]$, which implies that $\{x_n^{(1)}(t)\}$ and $\{p(t)(x_n^{(1)})'(t)\}$ are equi-continuous on $[0,1]$.

The Arzela-Ascoli theorem guarantees that there is a subsequence $\{x_{n_j}^{(1)}\}$ of $\{x_n^{(1)}\}$ with $\lim_{j\to+\infty} x_{n_j}^{(1)} = x_0^{(1)}$. From (3.6), we have

$$x_0(t) \geq k^*\gamma_1(t), \quad \forall t \in [0,1].$$

Then, for $t \in (0,1)$, if j is big enough, we have

$$\begin{aligned}
&|f_{n_j}(t, x_{n_j}^{(1)}(t), p(t)(x_{n_j}^{(1)})'(t)) - f(t, x_0^{(1)}(t), p(t)(x_0^{(1)})'(t))| \\
&= |f(t, x_{n_j}^{(1)}(t), p(t)(x_{n_j}^{(1)})'(t)) - f(t, x_0^{(1)}(t), p(t)(x_0^{(1)})'(t))| \to 0, \quad \text{as } j \to +\infty
\end{aligned}$$

and

$$\begin{aligned}
f_{n_j}(t, x_{n_j}^{(1)}(t), p(t)(x_{n_j}^{(1)})'(t)) &= f(t, \max\{\frac{1}{n_j}, x_{n_j}^{(1)}(t)\}, p(t)(x_{n_j}^{(1)})'(t)) \\
&\leq k(t)h(k^*\gamma_1(t))\frac{g(R_0)}{h(R_0)}.
\end{aligned} \tag{3.9}$$

Then the Lesbegue Dominated Convergence Theorem guarantees that

$$\begin{aligned}
x_0^{(1)}(t) &= \lim_{j\to+\infty} x_{n_j}^{(1)}(t) \\
&= \lim_{j\to+\infty} \int_0^1 G_1(t,s)\Phi(s)f(s, \max\{\frac{1}{n_j}, x_{n_j}^{(1)}(s)\}, p(s)(x_{n_j}^{(1)})'(s))ds \\
&= \int_0^1 G_1(t,s)\Phi(s)f(s, x_0^{(1)}(s), p(s)(x_0^{(1)})'(s))ds.
\end{aligned} \tag{3.10}$$

Obviously $\|x_0^{(1)}\|_1 \leq R_0$. Thus (3.3) can guarantee $\|x_0^{(1)}\|_1 < R_0$. Let $0 < a^* < b^* < 1$, and $0 < c^* < \min_{t\in[a^*,b^*]} \gamma_1(t)$. Suppose

$$N^* = \Big(\min_{t\in[a^*,b^*]} \int_{a^*}^{b^*} G_1(t,s)\Phi(s)k_1(s)dsc^*\Big)^{-1} + 1.$$

From the condition (H3), there exists an $R' > R$ such that

$$g_1(x) > N^*x, \forall x \geq R'. \tag{3.11}$$

Now we define

$$\Omega_2 = \{x \in \big|\|x\|_1 < \frac{R'}{c^*}\}. \tag{3.12}$$

We might as well suppose that $\frac{R'}{c^*} > 1$, $R' > 1$. Then we have

$$A_n x \nleq x \quad \text{for all } x \in \partial\Omega_2 \cap P,\ n \in \{1,2,\dots\}. \tag{3.13}$$

Otherwise, suppose there exists $x_0 \in \partial\Omega_2 \cap P$ with $A_n x_0 \leq x_0$. Since $x_0 \in \partial(\Omega_2) \cap P$,

$$\min_{t\in[a^*,b^*]} x_0(t) \geq \min_{t\in[a^*,b^*]} \gamma_1(t)\|x\|_1 > c^*\frac{R'}{c^*} = R' > 1. \tag{3.14}$$

Then, for $t \in [a^*,b^*]$, from (3.11) and (3.14), we have

$$\begin{aligned}
x_0(t) &\geq (A_n x_0)(t) \\
&= \int_0^1 G_1(t,s)\Phi(s)f_n(s, x_0(s), p(s)x_0'(s))ds \\
&\geq \int_{a^*}^{b^*} G_1(t,s)\Phi(s)f(s, \max\{\frac{1}{n}, x_0(s)\}, p(s)x_0'(s))ds
\end{aligned}$$

$$\begin{aligned}
&\geq \int_{a^*}^{b^*} G_1(t,s)\Phi(s)k_1(s)g_1(\max\{\frac{1}{n}, x_0(s)\})ds \\
&> \int_{a^*}^{b^*} G_1(t,s)\Phi(s)k_1(s)N^* x_0(s)ds \\
&> \int_{a^*}^{b^*} G_1(t,s)\Phi(s)k_1(s)dsN^* c^* \frac{R'}{c^*} \\
&> \frac{R'}{c^*}, \quad \forall n \in \{1,2,\dots\}
\end{aligned}$$

which implies $\|x_0\|_1 > R'/c^*$. This contradicts to $x_0 \in P \cap \partial\Omega_2$.

From (3.4) and (3.13), the Theorem 2.7 guarantees that A_n has a fixed point $x_n^{(2)} \in (\Omega_2 - \overline{\Omega}_1) \cap P$, $n \in \{1,2,\dots\}$. It is easy to see that

$$x_n^{(2)}(t) \geq \gamma_1(t)\|x_n^{(2)}\|_1 \geq \gamma_1(t)R_0, \quad n \in \{1,2,\dots\}. \tag{3.15}$$

By proof similar to (3.6), (3.7), (3.8), we know that $\{x_n^{(2)}\}$ is relatively compact in C_p^1. Then there exists a subsequence $\{x_{n_i}^{(2)}\}$ of $\{x_n^{(2)}\}$ with $\lim_{i\to+\infty} x_{n_i}^{(2)} = x_0^{(2)}$. And moreover, by similar proof as (3.10), we get $x_0^{(2)}(t)$ is a positive solution to equation (1.1) with $\frac{R'}{c^*} > \|x_0^{(2)}\|_1 > R_0$. Consequently, equation (1.1) has at least two different positive solution $x_0^{(1)}(t)$ and $x_0^{(2)}(t)$. The proof is complete. □

Example 3.4. Now we consider

$$\begin{gathered}
x'' + \frac{1}{16}t^{-1/2}(1-t)^{-1/4}(x^{-1/4} + x^2)(1 + \cos^2(x')) = 0, \quad t \in (0,1) \\
\lim_{t\to 0+} x'(t) = 0 = x(1).
\end{gathered} \tag{3.16}$$

Then (3.16) has at least two positive solutions.

To prove that (3.16) has at least two positive solutions, we apply Theorem 3.3 with $\Phi(t) = \frac{1}{16}t^{-1/2}(1-t)^{-1/4}$, $p(t) \equiv 1$, $f(t,x,z) = (x^{-1/4} + x^2)(1 + \cos^2(z))$, $k(t) \equiv 1$, $g(x) = 2(x^{-1/4} + x^2)$, $h(x) = x^{-1/4}$, $\gamma_1(t) = 1 - t$, $k_1(t) \equiv 1$, $g_1(x) = x^{-1/4} + x^2$, $\Psi_c(t) = c^{-1/4}$. It is easy to verify that (H1)–(H4) hold. Hence, (3.16) has at leat two positive solutions.

Example 3.5. Now we consider

$$\begin{gathered}
x'' + \frac{1}{12\pi}t^{-1/4}(1-t)^{-1/4}(x^{-1/4} + x^2)(\pi + \arctan x') = 0, \quad t \in (0,1) \\
x(0) = 0 = x(1).
\end{gathered} \tag{3.17}$$

Then (3.17) has at least two positive solutions.

To prove that (3.17) has at least two positive solutions, we apply Theorem 3.3 with $\Phi(t) = \frac{1}{12\pi}t^{-1/4}(1-t)^{-1/4}$, $p(t) \equiv 1$, $f(t,x,z) = (x^{-1/4} + x^2)(\pi + \arctan x')$, $k(t) \equiv 1$, $g(x) = \frac{3\pi}{2}(x^{-1/4} + x^2)$, $h(x) = x^{-1/4}$, $\gamma_1(t) = t(1-t)$, $k_1(t) \equiv 1$, $g_1(x) = x^{-1/4} + x^2$, $\Psi_c(t) = c^{-1/4}$. It is easy to verify that (H1)–(H4) hold. Hence, (3.17) has at leat two positive solutions.

4. The existence of multiple positive solutions to equation (1.2)

In this section, we consider (1.2) and suppose that $f \in C(R^+ \times R_0^+ \times R^+, R^+)$, $p \in C(R^+, R) \cap C(R_0^+, R_0^+) \cap C^1(R_0^+, R)$ with $\int_0^{+\infty} \frac{1}{p(r)} dr = +\infty$, $\Phi \in C(R_0^+, R^+)$; here $R^+ = [0, +\infty)$, $R_0^+ = (0, +\infty)$, $R = (-\infty, +\infty)$. Let

$$G_2(t,s) = \begin{cases} u_2(t)v_2(s)p(s), & a \le s \le t < +\infty \\ v_2(t)u_2(s)p(s), & 0 \le t \le s < +\infty, \end{cases}$$

where $u_2(t) = 1$ and $v_2(t) = \int_0^t \frac{1}{p(r)} dr$ for all $t \in R^+$.

Let $C_\infty^1 = \{x : [0, +\infty) \to R |\, x(t)$ is continuous on R^+ and $p(t)x'(t)$ is continuous on R^+ also with $\lim_{t\to+\infty} \frac{x(t)}{1+v_2(t)}$ exists and $\sup_{t\in[0,+\infty)} p(t)|x'(t)| < +\infty\}$. For $x \in C_\infty^1$, let

$$\|x\|_1 = \sup_{t\in[0,+\infty)} \frac{|x(t)|}{1 + v_2(t)} \quad \text{and} \quad \|x\|_2 = \sup_{t\in[0,+\infty)} p(t)|x'(t)|.$$

It is easy to see that $\|\cdot\|_1$ is a norm of C_∞^1 and $\|\cdot\|_2$ is a semi-norm of C_∞^1. Now Let $\|x\| = \max\{\|x\|_1, \|x\|_2\}$. Obviously, C_∞^1 satisfies (1), (2) and (3) of the Banach space E in section 2.

It is easy to prove that if $x(t) \in C_\infty^1$ is a solution to integral equation

$$x(t) = \int_0^\infty G_2(t,s)\Phi(s) f(s, x(s), p(s)x'(s)) ds, \quad t \in R^+,$$

then $x(t)$ is a solution to (1.2).

Let $P = \{x \in C_\infty^1 | x(t) \ge \gamma_2(t)\|x\|_1,\ \forall t \in R^+\}$, where

$$\gamma_2(t) = \begin{cases} \int_0^t \frac{1}{p(r)} dr, & t \in [0, \tau] \\ 1, & t \in (\tau, +\infty) \end{cases}$$

$$\widetilde{\gamma}_2(t) = \frac{\gamma_2(t)}{1 + v_2(t)}, \quad t \in R^+;$$

here $\int_0^\tau \frac{1}{p(r)} dr = 1$. Suppose that $x = (1 + v_2(t))y$, $t \in R^+$ and $F(t, y, z) = f(t, (1 + v_2(t))y, z) = f(t, x, z)$.

Now we will list some conditions for convenience:

(H1) There exists a $k \in C(R^+, R_0^+)$, a $g \in C(R_0^+, R_0^+)$ and a decreasing continuous function $h \in C(R_0^+, R_0^+)$ such that

$$F(t, y, z) \le k(t)g(y), \quad \forall (y, z) \in R_0^+ \times R_0^+, t \in R^+,$$

where $\frac{g(y)}{h(y)}$ is an increasing function and $\int_0^\infty p(s)\Phi(s)k(s)h(c\widetilde{\gamma}_2(s)) ds < +\infty$ for each $c > 0$

(H2)

$$\sup_{c\in R_0^+} \frac{ch(c)}{\int_0^\infty p(s)\Phi(s)k(s)h(c\widetilde{\gamma}_2(s)) ds\, g(c)} > 1$$

(H3) There exists a $k_1 \in C(R^+, R_0^+)$ and a $g_1 \in C(R_0^+, R_0^+)$ with $F(t, y, z) \ge k_1(t)g_1(y)$, for all $(t, y, z) \in [0, +\infty) \times R_0^+ \times R^+$ such that

$$\lim_{y\to+\infty} \frac{g_1(y)}{y} = +\infty,$$

where $\int_0^\infty p(s)\Phi(s)k_1(s) ds < +\infty$

(H4) for any $c > 0$, there exists a $\psi_c \in C(R^+, R_0^+)$ such that $F(t, y, z) \geq \psi_c(t)$ for all $(t, y, z) \in R^+ \times (0, c] \times R^+$ with $\int_0^\infty p(s)\Phi(s)\psi_c(s)ds < +\infty$.

Let $C_l = \{x : R^+ \to R|\ x(t) \text{ is continuous on } R^+ \text{ and } \lim_{t\to+\infty} x(t) \text{ exists }\}$ with norm $\|x\|_l = \sup_{t\in[0,+\infty)} |x(t)|$. From [18], we know that C_l is a Banach space and following theorem is true.

Theorem 4.1 ([18])**.** *Let $M \subseteq C_l(R^+, R)$. Then M is relatively compact in the space $C_l(R^+, R)$ if the following conditions hold:*

(a) *M is bounded in C_l*
(b) *the functions belonging to M are locally equi-continuous on R^+;*
(c) *the functions from M are equiconvergent, that is, given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|x(t) - x(+\infty)| < \varepsilon$ for any $t \geq T(\varepsilon)$ and $x \in M$.*

Theorem 4.2 ([22])**.** *Let $M \subseteq C^1_\infty(R^+, R)$. Then M is relatively compact in $C^1_\infty(R^+, R)$ if the following conditions hold:*

(a) *M is bounded in C^1_∞;*
(b) *the functions belonging to $\{y\big|y(t) = \frac{x(t)}{1+v_2(t)}, x \in M\}$ and the functions belonging to $\{y\big|y(t) = p(t)x'(t), x \in M\}$ are locally equi-continuous on R^+;*
(c) *the functions from $\{y\big|y(t) = \frac{x(t)}{1+v_2(t)}, x \in M\}$ and the functions from $\{y\big|y(t) = p(t)x'(t), x \in M\}$ are equi-convergent at $+\infty$.*

Lemma 4.3 ([22])**.** *Assume that $\overline{\Phi}(t) \in C(R_0^+, R^+)$ with $\int_0^{+\infty} p(s)\overline{\Phi}(t)dt < +\infty$ and $F(t) = \int_0^\infty G_2(t, s)\overline{\Phi}(s)ds$. Then*

$$F(t) \geq \gamma_2(t)\frac{F(\tau)}{1 + v_2(\tau)}, \quad \forall t \in R^+,\ \tau \in R^+,$$

and

$$F(t) \geq \gamma_2(t)\|F\|_1, \quad \forall t \in R^+,\ \lim_{t\to+\infty}\frac{F(t)}{1 + v_2(t)} = 0.$$

Let $f_n(t, x, z) = f(t, \max\{\frac{1}{n}(1 + v_2(t)), x\}, z)$, $n \in \{1, 2, \dots\}$ and for $x \in P$, $n \in \{1, 2, \dots\}$, $t \in R^+$, define

$$(A_n x)(t) = \int_0^\infty G_2(t, s)\Phi(s)f_n(s, x(s), p(s)x'(s))ds\,. \tag{4.1}$$

Lemma 4.4. *Assume that the condition (H1) holds. Then for each $n \in \{1, 2, \dots\}$, $A_n : P \to P$ is continuous and for any $r > 0$ and $B_r = \{x \in C^1_\infty|\|x\|_1 \leq r\}$, $A_n(P \cap B_r)$ is relatively compact for each $n \geq 1$.*

Proof. First, we show that $A_n P \subseteq P$. For any $x \in P$, we have

$$\begin{aligned}
|(A_n x)(t)| &= \Big|\int_0^\infty G_2(t, s)\Phi(s)f_n(s, x(s), p(s)x'(s))ds\Big| \\
&\leq \int_0^\infty G_2(t, s)\Phi(s)f(s, \max\{\frac{1}{n}(1 + v_2(s)), x(s)\}, p(s)x'(s))ds \\
&= \int_0^\infty G_2(t, s)\Phi(s)F(s, \max\{\frac{1}{n}, \frac{x(s)}{1 + v_2(s)}\}, p(s)x'(s))ds \\
&\leq \int_0^\infty G_2(t, s)\Phi(s)k(s)g(\max\{\frac{1}{n}, \frac{x(s)}{1 + v_2(s)}\})ds
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^\infty G_2(t,s)\Phi(s)k(s)h(\max\{\frac{1}{n},\frac{x(s)}{1+v_2(s)}\})\frac{g(\max\{\frac{1}{n},\frac{x(s)}{1+v_2(s)}\})}{h(\max\{\frac{1}{n},\frac{x(s)}{1+v_2(s)}\})}ds\\
&\le \int_0^\infty G_2(t,s)\Phi(s)k(s)ds h(\frac{1}{n})\frac{g(\max\{\frac{1}{n},\|x\|_1\})}{h(\max\{\frac{1}{n},\|x\|_1\})} < +\infty
\end{aligned}
$$

and

$$
\begin{aligned}
|p(t)(A_nx)'(t)| &= \big|\int_t^\infty p(s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds\big|\\
&\le \int_t^\infty p(s)\Phi(s)f(s,\max\{\frac{1}{n}(1+v_2(s)),x(s)\},p(s)x'(s))ds\\
&= \int_t^\infty p(s)\Phi(s)F(s,\max\{\frac{1}{n},\frac{x(s)}{1+v_2(s)}\},p(s)x'(s))ds\\
&\le \int_t^\infty p(s)\Phi(s)k(s)g(\max\{\frac{1}{n},\frac{x(s)}{1+v_2(s)}\})ds\\
&\le \int_t^\infty p(s)\Phi(s)k(s)h(\max\{\frac{1}{n},\frac{x(s)}{1+v_2(s)}\})\frac{g(\max\{\frac{1}{n},\frac{x(s)}{1+v_2(s)}\})}{h(\max\{\frac{1}{n},\frac{x(s)}{1+v_2(s)}\})}ds\\
&\le \int_0^\infty p(s)\Phi(s)k(s)ds h(\frac{1}{n})\frac{g(\max\{\frac{1}{n},\|x\|_1\})}{h(\max\{\frac{1}{n},\|x\|_1\})}\\
&< +\infty.
\end{aligned}
$$

Since $(A_nx)(t) \ge 0$, it is easy to see that A_n is well defined. Moreover, from Lemma 4.3, for any $x \in P$, we have

$$(A_nx)(t) \ge \gamma_2(t)\|A_nx\|_1, \quad \forall t \in R^+.$$

Consequently, $A_nP \subseteq P$.

Second, we show $A_n : P \to P$ is continuous. Assume that $\lim_{m\to+\infty} x_m = x_0$, which means there exists an $M > 1/n$ such that $\|x_m\| \le M$, for all $m \in \{0, 1, 2, \dots\}$. Then

$$
\begin{aligned}
f_n(t,x_m(t),p(t){x'}_m(t)) &= F(t,\max\{\frac{1}{n},\frac{x_m(t)}{1+v_2(t)}\},p(t)x'_m(t))\\
&\le k(t)g(\max\{\frac{1}{n},\frac{x_m(t)}{1+v_2(t)}\})\\
&\le k(t)h(\max\{\frac{1}{n},\frac{x_m(t)}{1+v_2(t)}\})\frac{g(\max\{\frac{1}{n},\frac{x_m(t)}{1+v_2(t)}\})}{h(\max\{\frac{1}{n},\frac{x_m(t)}{1+v_2(t)}\})}\\
&\le k(t)h(\frac{1}{n})\frac{g(M)}{h(M)}, \quad m \in \{1,2,\dots\}.
\end{aligned}
\tag{4.2}
$$

Since

$$f_n(t,x_m(t),p(t)x'_m(t)) \to f_n(t,x_0(t),p(t)x'_0(t)), \quad \text{as } m \to +\infty,$$

from (4.2), the Lesbegue Dominated Convergence Theorem guarantees that

$$\lim_{m\to+\infty}\sup_{t\in[0,+\infty)}\frac{|(A_nx_m)(t)-(A_nx_0)(t)|}{1+v_2(t)}$$

$$
\begin{aligned}
&= \lim_{m\to+\infty} \sup_{t\in[0,+\infty)} \Big| \int_0^\infty \frac{G_2(t,s)}{1+v_2(t)} \Phi(s)(f_n(s,x_m(s),p(s)x_m'(s)) \\
&\quad - f_n(s,x_0(s),p(s)x_0'(s)))ds\Big| \\
&\le \lim_{m\to+\infty} \sup_{t\in[0,+\infty)} \int_0^\infty \frac{G_2(t,s)}{1+v_2(t)} \Phi(s)|f_n(s,x_m(s),p(s)x_m'(s)) \\
&\quad - f_n(s,x_0(s),p(s)x_0'(s))|ds \\
&= \lim_{m\to+\infty} \int_0^\infty p(s)\Phi(s)|f_n(s,x_m(s),p(s)x_m'(s)) \\
&\quad - f_n(s,x_0(s),p(s)x_0'(s))|ds = 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{m\to+\infty} \sup_{t\in[0,+\infty)} |p(t)(A_nx_m)'(t) - p(t)(A_nx_0)'(t)| \\
&= \lim_{m\to+\infty} \sup_{t\in[0,+\infty)} \Big| \int_t^\infty p(s)\Phi(s)f_n(s,x_m(s),p(s)x_m'(s))ds \\
&\quad - \int_t^\infty p(s)\Phi(s)f_n(s,x_0(s),p(s)x_0'(s))ds\Big| \\
&\le \lim_{m\to+\infty} \sup_{t\in[0,+\infty)} \int_t^\infty p(s)\Phi(s)|f_n(s,x_m(s),p(s)x_m'(s)) \\
&\quad - f_n(s,x_0(s),p(s)x_0'(s))|ds \\
&= \lim_{m\to+\infty} \int_0^\infty p(s)\Phi(s)\big|f_n(s,x_m(s),p(s)x_m'(s)) - f_n(s,x_0(s),p(s)x_0'(s))\big|ds = 0,
\end{aligned}
$$

which implies

$$
\lim_{m\to+\infty} \|A_nx_m - A_nx_0\| = 0.
$$

Finally, we show $A_n(B_r\cap P)$ is relatively compact. Obviously, B_r is a unbounded set in C^1_∞. Without loss of generality, we suppose $r > 1/n$. Then, for any $x \in B_r \cap P$, we have

$$
\begin{aligned}
&\sup_{t\in[0,+\infty)} \frac{|(A_nx)(t)|}{1+v_2(t)} \\
&= \sup_{t\in[0,+\infty)} \Big| \int_0^\infty \frac{G_2(t,s)}{1+v_2(t)} \Phi(s)f_n(s,x(s),p(s)x'(s))ds\Big| \\
&= \sup_{t\in[0,+\infty)} \int_0^\infty \frac{G_2(t,s)}{1+v_2(t)} \Phi(s)f_n(s,x(s),p(s)x'(s))ds \\
&\le \sup_{t\in[0,+\infty)} \int_0^\infty \frac{G_2(t,s)}{1+v_2(t)} \Phi(s)k(s)h(\max\{\frac{1}{n}, \frac{x(s)}{1+v_2(s)}\}) \frac{g(\max\{\frac{1}{n}, \frac{x(s)}{1+v_2(s)}\})}{h(\max\{\frac{1}{n}, \frac{x(s)}{1+v_2(s)}\})} ds \\
&\le \sup_{t\in[0,+\infty)} \int_0^\infty \frac{G_2(t,s)}{1+v_2(t)} \Phi(s)k(s)ds h(\frac{1}{n}) \frac{g(r)}{h(r)} \\
&\le \int_0^\infty p(s)\Phi(s)k(s)ds h(\frac{1}{n}) \frac{g(r)}{h(r)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\sup_{t\in[0,+\infty)}|p(t)(A_nx)'(t)| \\
&= \sup_{t\in[0,+\infty)}|\int_t^\infty p(s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds| \\
&= \sup_{t\in[0,+\infty)}\int_t^\infty p(s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds \\
&\le \sup_{t\in[0,+\infty)}\int_t^\infty p(s)\Phi(s)k(s)h(\max\{\frac{1}{n},\frac{x(s)}{1+v_2(s)}\})\frac{g(\max\{\frac{1}{n},\frac{x(s)}{1+v_2(s)}\})}{h(\max\{\frac{1}{n},\frac{x(s)}{1+v_2(s)}\})}ds \\
&\le \sup_{t\in[0,+\infty)}\int_t^\infty p(s)\Phi(s)k(s)dsh(\frac{1}{n})\frac{g(r)}{h(r)} \\
&= \int_0^\infty p(s)\Phi(s)k(s)dsh(\frac{1}{n})\frac{g(r)}{h(r)},
\end{aligned}
$$

which implies $A_n(P\cap B_r)$ is bounded. Assume that $t,t'\in R^+$. Then, for $x\in B_r\cap P$, we have

$$
\begin{aligned}
&|\frac{(A_nx)(t)}{1+v_2(t)}-\frac{(A_nx)(t')}{1+v_2(t')}| \\
&= |\int_0^\infty \frac{G_2(t,s)}{1+v_2(t)}\Phi(s)f_n(s,x(s),p(s)x'(s))ds \\
&\quad -\int_0^\infty \frac{G_2(t',s)}{1+v_2(t')}\Phi(s)f_n(s,x(s),p(s)x'(s))ds| \\
&\le \int_0^\infty |\frac{G_2(t,s)}{1+v_2(t)}-\frac{G_2(t',s)}{1+v_2(t')}|\Phi(s)f_n(s,x(s),p(s)x'(s))ds \\
&\le \int_0^\infty |\frac{G_2(t,s)}{1+v_2(t)}-\frac{G_2(t',s)}{1+v_2(t')}|\Phi(s)k(s)dsh(\frac{1}{n})\frac{g(r)}{h(r)}
\end{aligned}
$$

and

$$
\begin{aligned}
&|p(t)(A_nx)'(t)-p(t')(A_nx)'(t')| \\
&= \Big|\int_t^1 p(s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds-\int_{t'}^1 p(s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds\Big| \\
&= \Big|\int_t^{t'} p(s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds\Big| \\
&\le \Big|\int_t^{t'} p(s)\Phi(s)k(s)ds\Big|h(\frac{1}{n})\frac{g(r)}{h(r)}.
\end{aligned}
$$

For any $\varepsilon>0$, $T>0$, we can choose $\delta>0$ small enough such that

$$
|\frac{(A_nx)(t)}{1+v_2(t)}-\frac{(A_nx)(t')}{1+v_2(t')}|<\varepsilon,\quad |p(t)(A_nx)'(t)-p(t')(A_nx)'(t')|<\varepsilon,
$$

for all $x\in B_r\cap P$, $|t-t'|<\delta$, $t,t'\in[0,T]$. Consequently, $\{\frac{(A_n(B_r\cap P))(t)}{1+v_2(t)}\}$ and $\{p(t)(A_n(B_r\cap P))'(t)\}$ is locally equi-continuous on $[0,+\infty)$.

Moreover, Lemma 4.3 guarantees that

$$\begin{aligned}
&\lim_{t\to+\infty}\sup_{x\in P\cap B_r}\Big|\frac{(Ax)(t)}{1+v_2(t)}\Big|\\
&\leq \lim_{t\to+\infty}\int_0^\infty \frac{G_2(t,s)}{1+v_2(t)}\Phi(s)f_n(s,x(s),p(s)x'(s))ds\\
&\leq \lim_{t\to+\infty}\sup_{x\in P\cap B_r}\int_0^\infty \frac{G_2(t,s)}{1+v_2(t)}\Phi(s)k(s)g(\max\{\frac{1}{n},\frac{x(s)}{1+v_2(s)}\})ds\\
&\leq \lim_{t\to+\infty}\frac{\int_0^\infty G_2(t,s)\Phi(s)k(s)ds}{1+v_2(t)}h(\frac{1}{n})\frac{g(r)}{h(r)}=0
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to+\infty}\sup_{x\in P\cap B_r}|p(t)(Ax)'(t)| &\leq \lim_{t\to+\infty}\int_t^\infty p(s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds\\
&\leq \lim_{t\to+\infty}\sup_{x\in P\cap B_r}\int_t^\infty p(s)\Phi(s)k(s)g(\max\{\frac{1}{n},x(s)\})ds\\
&\leq \lim_{t\to+\infty}\int_t^\infty p(s)\Phi(s)k(s)dsh(\frac{1}{n})\frac{g(r)}{h(r)}=0,
\end{aligned}$$

which imply that the functions belonging to $\{\frac{(A(P\cap B_r))(t)}{1+v_2(t)}\}$ and the functions belonging to $\{p(t)(A(P\cap B_r))'(t)\}$ are equi-convergent.

As a result, from Lemma 4.3, $A_n(B_r\cap P)$ is relatively compact. The proof is complete. □

Theorem 4.5. *Assume that (H1)–(H4) hold. Then* (1.2) *has at least two positive solutions.*

Proof. From Lemma 4.4, for each $n\in\{1,2,\dots\}$, $A_n:P\to P$ is continuous operator and for any $r>0$, $A_n(B_r\cap P)$ is relatively compact. From (H2), choose $R_0>0$ such that

$$\frac{R_0h(R_0)}{\int_0^\infty p(s)\Phi(s)k(s)h(R_0\tilde{\gamma}_2(s))dsg(R_0)}>1. \tag{4.3}$$

Without loss of the generality, suppose $R_0\geq 1/n$. Set

$$\Omega_1=\{x\in C^1_\infty|\|x\|_1<R_0\}. \tag{4.4}$$

Then for any $x\in\partial\Omega_1\cap P$, one has

$$x\not\leq A_nx,\quad n\in\{1,2,\dots\}. \tag{4.5}$$

If there exists $x_0\in\partial\Omega_1\cap P$ such that $x_0\leq A_nx_0$, obviously, $\max\{\frac{1}{n},x_0(t)\}\geq x_0(t)\geq\gamma_2(t)R_0$ and $\frac{\max\{\frac{1}{n},x_0(t)\}}{1+v_2(t)}\geq\tilde{\gamma}_2(t)R_0$. Then

$$\begin{aligned}
R_0&=\sup_{t\in[0,+\infty)}\frac{|x_0(t)|}{1+v_2(t)}\\
&\leq \sup_{t\in[0,+\infty)}\int_0^\infty\frac{G_2(t,s)}{1+v_2(t)}\Phi(s)f(s,\max\{\frac{1}{n},x_0(s)\},p(s){x'}_0(s))ds\\
&\leq \int_0^\infty p(s)\Phi(s)k(s)g(\frac{\max\{\frac{1}{n},\frac{x_0(s)}{1+v_2(s)}\}}{1+v_2(s)})ds
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^{\infty} p(s)\Phi(s)k(s)h(\max\{\frac{1}{n}, \frac{x_0(s)}{1+v_2(s)}\})\frac{g(\max\{\frac{1}{n}, \frac{x_0(s)}{1+v_2(s)}\})}{h(\max\{\frac{1}{n}, \frac{x_0(s)}{1+v_2(s)}\})}ds \\
&\le \int_0^{\infty} p(s)\Phi(s)k(s)h(R_0\tilde{\gamma}_2(s))ds\frac{g(R_0)}{h(R_0)},
\end{aligned}
$$

which implies

$$
\frac{R_0 h(R_0)}{\int_0^{+\infty} p(s)\Phi(s)k(s)h(R_0\tilde{\gamma}_2(s))ds g(R_0)} \le 1.
$$

This contradicts (4.3). Then (4.5) is true. From the proof of Theorem 2.7, the (2.13) is true, which means

$$
i(A_n, P\cap\Omega_1, P) = 1, \quad n\in\{1,2,\dots\}. \tag{4.6}
$$

So for any $n \ge 1$, there exists an $x_n^{(1)} \in P\cap\Omega_1$ with $A_n x_n^{(1)} = x_n^{(1)}$.

From $\{x_n^{(1)}\} \subseteq \Omega_1 \cap P$, the $\{x_n^{(1)}\}$ is bounded. From Lemma 4.3, the condition (H4), there exists a $\psi_{R_0}(t)$ such that

$$
\begin{aligned}
x_n^{(1)}(t) &= \int_0^{\infty} G_2(t,s)\Phi(s)f(s,\max\{\frac{1}{n}, x_n^{(1)}(s)\}, p(s)(x_n^{(1)})'(s))ds \\
&\ge \int_0^{\infty} G_2(t,s)\Phi(s)\psi_{R_0}(s)ds \\
&\ge \gamma_2(t)k^*,
\end{aligned} \tag{4.7}
$$

where $k^* = \sup_{t\in[0,+\infty)} \int_0^{\infty} \frac{G_2(t,s)}{1+v_2(t)}\Phi(s)\psi_{R_0}(s)ds$. Then, from Lemma 4.3, one has

$$
\begin{aligned}
&\lim_{t\to+\infty}\sup_{n\ge1} \Big|\frac{x_n^{(1)}(t)}{1+v_2(t)}\Big| \\
&\le \lim_{t\to+\infty} \int_0^{\infty} \frac{G_2(t,s)}{1+v_2(t)}\Phi(s)f_n(s,x(s),p(s)x'(s))ds \\
&\le \lim_{t\to+\infty}\sup_{n\ge1} \int_0^{\infty} \frac{G_2(t,s)}{1+v_2(t)}\Phi(s)k(s)g(\max\{\frac{1}{n}, \frac{x_n^{(1)}(s)}{1+v_2(s)}\})ds \\
&\le \lim_{t\to+\infty} \frac{\int_0^{\infty} G_2(t,s)\Phi(s)k(s)h(k^*\tilde{\gamma}_2(s))ds}{1+v_2(t)}\frac{g(R_0)}{h(R_0)} = 0
\end{aligned} \tag{4.8}
$$

and

$$
\begin{aligned}
&\lim_{t\to+\infty}\sup_{n\ge1} |p(t)(x_n^{(1)})'(t)| \\
&\le \lim_{t\to+\infty}\sup_{n\ge1} \int_t^{\infty} p(s)\Phi(s)f_n(s,x(s),p(s)x'(s))ds \\
&\le \lim_{t\to+\infty}\sup_{n\ge1} \int_t^{\infty} p(s)\Phi(s)k(s)g(\max\{\frac{1}{n}, \frac{x_n^{(1)}(s)}{1+v_2(s)}\})ds \\
&\le \lim_{t\to+\infty} \int_t^{\infty} p(s)\Phi(s)k(s)h(k^*\tilde{\gamma}_2(s))ds\frac{g(R_0)}{h(R_0)} \\
&= 0,
\end{aligned} \tag{4.9}
$$

which imply that the functions belonging to $\{\frac{x_n^{(1)}(t)}{1+v_2(t)}\}$ and the functions belonging to $\{p(t)(x_n^{(1)})'(t)\}$ are equi-convergent. Moreover, for any $t', t'' \in [0,+\infty)$, $n \in$

$\{1, 2, \dots\}$,

$$\begin{aligned} &|\frac{x_n^{(1)}(t')}{1+v_2(t')} - \frac{x_n^{(1)}(t'')}{1+v_2(t'')}| \\ &\leq \int_0^\infty |\frac{G_2(t',s)}{1+v_2(t')} - \frac{G_2(t'',s)}{1+v_2(t'')}| f_n(s, x_n^{(1)}(s), p(s)(x_n^{(1)})'(s)) ds \\ &\leq \int_0^\infty |\frac{G_2(t',s)}{1+v_2(t')} - \frac{G_2(t'',s)}{1+v_2(t'')}| \Phi(s) k(s) h(\tilde{\gamma}_2(s) k^*) ds \frac{g(R_0)}{h(R_0)}, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} &|p(t')(x_n^{(1)})'(t') - p(t'')(x_n^{(1)})'(t'')| \\ &\leq |\int_{t'}^{t''} p(s)\Phi(s) f_n(s, x_n^{(1)}(s), p(s)(x_n^{(1)})'(s)) ds| \\ &\leq |\int_{t'}^{t''} p(s)\Phi(s) k(s) h(\tilde{\gamma}_2(s) k^*) ds| \frac{g(R_0)}{h(R_0)}, \quad n \in \{1, 2, \dots\}. \end{aligned} \tag{4.11}$$

Consequently, for any $\varepsilon > 0$, $T > 0$, we can choose a $\delta > 0$ such that if $|t' - t''| < \delta$ and t', $t'' \in [0, T]$, then

$$|\frac{x_n^{(1)}(t')}{1+v_2(t')} - \frac{x_n^{(1)}(t'')}{1+v_2(t'')}| < \varepsilon, \quad |p(t')(x_n^{(1)})'(t') - p(t'')(x_n^{(1)})'(t'')| < \varepsilon,$$

for all $n \in \{1, 2, \dots\}$, which implies that $\{\frac{x_n^{(1)}(t)}{1+v_2(t)}\}$ and $\{p(t)(x_n^{(1)})'(t)\}$ are locally equi-continuous on $[0, +\infty)$. Thus, Theorem 4.2 guarantees that there is a convergent subsequence $\{x_{n_j}^{(1)}\}$ of $\{x_n^{(1)}\}$ with $\lim_{j\to+\infty} x_{n_j}^{(1)} = x_0^{(1)}$. From (4.7), we have

$$x_0^{(1)}(t) \geq k^* \gamma_2(t), \quad \forall t \in [0, +\infty).$$

Then, for $t \in (0, +\infty)$, if j is big enough, we have

$$\begin{aligned} &|f_{n_j}(t, x_{n_j}^{(1)}(t), p(t)(x_{n_j}^{(1)})'(t)) - f(t, x_0^{(1)}(t), p(t)(x_0^{(1)})'(t))| \\ &= |f(t, x_{n_j}^{(1)}(t), p(t)(x_{n_j}^{(1)})'(t)) - f(t, x_0^{(1)}(t), p(t)(x_0^{(1)})'(t))| \to 0, \end{aligned}$$

as $j \to +\infty$, and

$$\begin{aligned} f_{n_j}(t, x_{n_j}^{(1)}(t), p(t)(x_{n_j}^{(1)})'(t)) &= f(t, \max\{\frac{1}{n_j}(1+v_2(t)), x_{n_j}^{(1)}(t)\}, p(t)(x_{n_j}^{(1)})'(t)) \\ &\leq k(t) h(k^* \tilde{\gamma}_2(t)) \frac{g(R_0)}{h(R_0)}. \end{aligned}$$

Then the Lesbegue Dominated Convergence Theorem guarantees that

$$\begin{aligned} &x_0^{(1)}(t) \\ &= \lim_{j\to+\infty} x_{n_j}^{(1)}(t) \\ &= \lim_{j\to+\infty} \int_0^\infty G_2(t,s)\Phi(s) f(s, \max\{\frac{1}{n_j}(1+v_2(s)), x_{n_j}^{(1)}(s)\}, p(s)(x_{n_j}^{(1)})'(s)) ds \\ &= \int_0^\infty G_2(t,s)\Phi(s) f(s, x_0^{(1)}(s), p(s)(x_0^{(1)})'(s)) ds, \end{aligned} \tag{4.12}$$

which implies $x_0^{(1)}(t)$ is a positive solution to (1.2). Obviously, $\|x_0^{(1)}\|_1 \le R_0$. Then (4.3) can guarantee $\|x_0^{(1)}\|_1 < R_0$.

Let $\tau < a^* < b^* < +\infty$, and $0 < c^* < \min_{t\in[a^*,b^*]} \tilde{\gamma}_2(t)$. Suppose

$$N^* = \Big(\min_{t\in[a^*,b^*]} \int_{a^*}^{b^*} \frac{G_2(t,s)}{1+v_2(t)} \Phi(s)k_1(s)dsc^* \Big)^{-1} + 1.$$

From condition (H2), there exists an $R' > R$ such that

$$g_1(y) > N^* y, \quad \forall y \ge R'. \tag{4.13}$$

Now we define

$$\Omega_2 = \{x \in C^1_\infty \big| \|x\|_1 < \frac{R'}{c^*}\}. \tag{4.14}$$

We might as well suppose that $\frac{R'}{c^*} > 1$, $R' > 1$. Then we have

$$A_n x \not\le x, \quad n \in \{1,2,\dots\} \tag{4.15}$$

for all $x \in \partial\Omega_2 \cap P$. Otherwise, suppose there exists $x_0 \in \partial\Omega_2 \cap P$ with $A_n x_0 \le x_0$. Since $x_0 \in \partial(\Omega_2) \cap P$,

$$\min_{t\in[a^*,b^*]} \frac{x_0(t)}{1+v_2(t)} \ge \min_{t\in[a^*,b^*]} \tilde{\gamma}_2(t) \sup_{t\in[0,+\infty)} \frac{|x(t)|}{1+v_2(t)} > c^* \frac{R'}{c^*} = R' > 1.$$

Then for $t \in [a^*, b^*]$, from (4.13), we have

$$\begin{aligned}
\frac{x_0(t)}{1+v_2(t)} &\ge \frac{(A_n x_0)(t)}{1+v_2(t)} \\
&= \int_0^{+\infty} \frac{G_2(t,s)}{1+v_2(t)} \Phi(s) f_n(s, x_0(s), p(s)x_0'(s))ds \\
&\ge \int_{a^*}^{b^*} \frac{G_2(t,s)}{1+v_2(t)} \Phi(s) f(s, \max\{\frac{1}{n}(1+v_2(s)), x_0(s)\}, p(s)x_0'(s))ds \\
&\ge \int_{a^*}^{b^*} \frac{G_2(t,s)}{1+v_2(t)} \Phi(s)k_1(s) g_1(\max\{\frac{1}{n}, \frac{x_0(s)}{1+v_2(s)}\})ds \\
&> \int_{a^*}^{b^*} \frac{G_2(t,s)}{1+v_2(t)} \Phi(s)k_1(s) N^* \frac{x_0(s)}{1+v_2(s)} ds \\
&> \int_{a^*}^{b^*} G_2(t,s)\Phi(s)k_1(s)dsN^* c^* \frac{R'}{c^*} \\
&> \frac{R'}{c^*},
\end{aligned}$$

which implies $\|x_0\|_1 > R'/c^*$. This contradicts to $x_0 \in P \cap \partial\Omega_2$. Then (4.15) is true.

From (4.5) and (4.15), Theorem 2.7 guarantees that A_n has a fixed point $x_n^{(2)} \in (\Omega_2 - \overline{\Omega}_1) \cap P$. For the set $\{x_n^{(2)}\}$, since $\|x_n^{(2)}\|_1 = \sup_{t\in[0,+\infty)} \frac{|x_n^{(2)}(t)|}{1+v_2(t)} \le \frac{R'}{c^*}$, (H4) guarantees that there is a $\psi_{R'/c^*}(t)$ such that

$$f(t,x,z) = F(t,y,z) \ge \psi_{\frac{R'}{c^*}}(t), \quad \forall (t,y,z) \in [0,+\infty) \times (0, \frac{R'}{c^*}] \times [0,+\infty). \tag{4.16}$$

By proof as in (4.7), (4.8), (4.9), (4.10) and (4.11), we can prove that $\{x_n^{(2)}\}$ is relatively compact in C^1_∞, which means that there exists a subsequence $\{x_{n_i}^{(2)}\}$ of

$\{x_n^{(2)}\}$ with $\lim_{i\to+\infty} x_{n_i}^{(2)} = x_0^{(2)}$. By proof as in (4.12) $x_0^{(2)}(t)$ is a positive solution to equation (1.2) with $\frac{R'}{c^*} > \|x_0^{(2)}\|_1 > R_0$.

Consequently, equation (1.2) has at least two different positive solutions $x_0^{(1)}(t)$ and $x_0^{(2)}(t)$. The proof is complete. □

Example 4.6. Now we consider

$$\begin{gathered} x'' + \frac{1}{16}e^{-t}t^{-1/4}((1+t)^{1/2}x^{-1/2} + \frac{1}{(1+t)^3}x^3)(1+\frac{x'^2}{1+x'^2}) = 0,\ t\in(0,+\infty) \\ x(0)=0, \quad \lim_{t\to+\infty} x'(t) = 0. \end{gathered} \tag{4.17}$$

Then, equation (4.17) has at least two positive solutions.

To prove that (4.17) has at least two positive solutions, we apply Theorem 4.5 with $\Phi(t) = \frac{1}{16}e^{-t}t^{-1/4}$, $p(t)\equiv 1$, $f(t,x,z) = ((1+t)^{1/2}x^{-1/2} + \frac{1}{(1+t)^3}x^3)(1+\frac{x'^2}{1+x'^2})$, $k(t)\equiv 1$, $g(y) = 2(y^{-1/2}+y^3)$, $h(x) = y^{-1/2}$, $\gamma_1(t) = \begin{cases} t, & t\in[0,1] \\ 1, & t\in(0,+\infty) \end{cases}$, $\tilde\gamma_2(t) = \frac{\gamma_2(t)}{1+t}$, $k_1(t)\equiv 1$, $g_1(y) = y^{-1/4}+y^3$, $\Psi_c(t) = c^{-1/4}$. It is easy to verify that (H1)–(H4) hold. Hence, (4.17) has at leat two positive solutions.

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