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WEAK SOLUTIONS FOR A STRONGLY-COUPLED NONLINEAR SYSTEM

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ABSTRACT. In this paper the authors study the existence of local weak solutions of the strongly nonlinear system

$$u'' + \mathcal{A}u + f(u,v)u = h_1$$
$$v'' + \mathcal{A}v + g(u,v)v = h_2$$

where $\mathcal{A}$ is the pseudo-Laplacian operator and f, g, h_1 and h_2 are given functions.

1. Introduction

Let Ω be an open and bounded subset in $\mathbb{R}^n$ with smooth boundary Γ and let T be a positive real number. In the cylinder $Q = \Omega\times]0,T[$, with lateral boundary $\sum = \Gamma\times]0,T[$, we consider the nonlinear system

$$\begin{gathered} u'' + \mathcal{A}u + f(u,v)u = h_1 \\ v'' + \mathcal{A}v + g(u,v)v = h_2 \\ u(0) = u_0, \quad v(0) = v_0, \quad u'(0) = u_1, \quad v'(0) = v_1 \\ u = v = 0 \quad \text{on } \Sigma = \Gamma\times]0,T[\end{gathered} \tag{1.1}$$

where

$$\mathcal{A}u = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(|\frac{\partial u}{\partial x_i}|^{p-2}\frac{\partial u}{\partial x_i}\Big), \quad p > 2,$$

is the pseudo-Laplacian operator, f is a continuous function in the first variable and Lipschitz in the second variable and g is a Lipschitz's function in the first variable and continuous in the second variable, with $f(0,0) = g(0,0) = 0$ and u_0, v_0, u_1, v_1, h_1 and h_2 are given functions.

When $p \geq 2$, many authors studied the system (1.1). For instance, we can mention: Segal [11], where the physical meaning of (1.1) is presented, Medeiros and Menzala [9], Medeiros and M. Miranda [10], Castro [3], Biazutti [1] and more recently, Clark and Lima [6] showed the existence, a local solution and its uniqueness for the system

$$u'' - \Delta u + f(u,v)u = h_1 \quad \text{in } Q = \Omega \times (0,T)$$

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$$
\begin{gathered}
v'' - \Delta u + g(u,v)v = h_2 \quad \text{in } Q \\
u(0) = u_0, \quad u'(0) = u_1 \quad \text{in } \Omega \\
v(0) = v_0, \quad v'(0) = v_1 \quad \text{in } \Omega \\
u = 0, \quad v = 0 \quad \text{on } \Sigma = \Gamma \times (0,T),
\end{gathered}
$$

where the functions f and g satisfying the same conditions of the problem (1.1). Castro [3] showed the existence of solution for the system

$$
\begin{gathered}
u'' + \mathcal{A}u - \Delta u' + |v|^{\rho+2}|u|^{\rho}u = f_1 \quad \text{in } Q \\
v'' + \mathcal{A}v - \Delta v' + |u|^{\rho+2}|v|^{\rho}v = f_2 \quad \text{in } Q \\
u(0) = u_0, \quad u'(0) = u_1 \quad \text{in } \Omega \\
v(0) = v_0, \quad v'(0) = v_1 \quad \text{in } \Omega \\
u = 0, \quad v = 0 \quad \text{on } \Sigma,
\end{gathered}
$$

where $\mathcal{A}$ is the pseudo-Laplacian operator. We can show that the functions $f(u,v) = |u|^{\rho+2}|v|^{\rho}$ and $g(u,v) = |v|^{\rho+2}|u|^{\rho}, \rho \geq -1$, satisfy the conditions of the system (1.1). Consequently the above system, without the dissipations $\Delta u'$ and $\Delta v'$, is a particular case of $(*)$. Thus, we see that (1.1) generalizes the above mentioned problems.

To show the existence of a *local* solution for (1.1), we encounter following technical difficulties:

(i) The choices of the functional spaces;
(ii) In the a priori estimate for u''_m, we had that to use the projection operator, since, to derive the approximated equation we will have much technical difficulties because of the pseudo-Laplacian operator in the equation;
(iii) In the passage to the limit, we use strongly the fact that $\mathcal{A}$ is a monotonic and hemicontinuous operator.

We remark that these difficulties do not appear in [6].

Notation. We represent the Sobolev space of order m in Ω by

$$
W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \forall |\alpha| \leq m\},
$$

with the norm

$$
\|u\|_{m,p} = \Big(\sum_{|\alpha| \leq m} |D^\alpha u|^p_{L^p(\Omega)} \Big)^{1/p}, u \in W^{m,p}(\Omega), 1 \leq p < \infty.
$$

Let $\mathcal{D}(\Omega)$ be the space of test functions in Ω and by $W_0^{m,p}(\Omega)$ we represent the closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$. The dual space of $W_0^{m,p}(\Omega)$ is denoted by $W^{-m,p'}(\Omega)$ with p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$. We use the symbols $(\cdot,\cdot)$ and $|\cdot|$, to indicate the inner product and the norm in $L^2(\Omega)$. We use $\langle \cdot,\cdot \rangle_{W^{-1,p}(\Omega), W_0^{1,p}(\Omega)}$ to indicate the duality between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$ and $\|\cdot\|_0$ to indicate the norm $W_0^{1,p}(\Omega)$. The pseudo-Laplacian operator $\mathcal{A}$ is such that

$$
\begin{aligned}
\mathcal{A} : \quad W_0^{1,p}(\Omega) &\rightarrow W^{-1,p'}(\Omega) \\
u &\mapsto \mathcal{A}u
\end{aligned}
$$

and it satisfies the following properties:

- $\mathcal{A}$ is monotonic, that is, $\langle \mathcal{A}u - \mathcal{A}v, u - v \rangle \geq 0, \forall u, v \in W_0^{1,p}(\Omega)$;

- $\mathcal{A}$ is hemicontinuous, that is, for each $u, v, w \in W_0^{1,p}(\Omega)$ the function $\lambda \mapsto \langle \mathcal{A}(u + \lambda v), w\rangle$ is continuous in $\mathbb{R}$;
- $\langle \mathcal{A}u(t), u(t)\rangle_{W^{-1,p'}(\Omega)\times W_0^{1,p}(\Omega)} = \|u\|_0^p$;
- $\langle \mathcal{A}u(t), u'(t)\rangle_{W^{-1,p'}(\Omega)\times W_0^{1,p}(\Omega)} = \frac{1}{p}\frac{d}{dt}\|u\|_0^p$, $\frac{d}{dt} ='$;
- $\|\mathcal{A}u(t)\|_{W^{-1,p'}(\Omega)} \le C\|u\|_0^{p-1}$, where C is a constant;

We will use the same notation for the operator P and its restrictions, as well as for the operator P^*.

The next lemma plays a central role in the proof of the Existence Theorem. Its proof can be found in [6].

Lemma 1.1. *Let ϕ be a positive real function, α, β and γ, positive real constants, with $\gamma > 1$, such that*

$$\phi(t) \le \alpha + \beta \int_0^t \big\{\phi(s) + \phi^{\gamma}(s)\big\} ds.$$

Then, there exists $T_0 \in \mathbb{R}$, with $0 < T_0 < T$, such that ϕ is bounded in $[0, T_0[$.

Definition. A local weak solution of the problem (1.1) is a pair of functions $u = u(x,t)$, $v = v(x,t)$ defined for all $(x,t) \in Q_{T_0} = \Omega \times (0, T_0)$, and $T_0 > 0$ fixed, satisfying

$$u, v \in L^\infty(0, T_0; W_0^{1,p}(\Omega));$$
$$u', v' \in L^\infty(0, T_0; L^2(\Omega));$$
$$\frac{d}{dt}(u', w) + \langle \mathcal{A}u, w\rangle + \langle f(u,v)u, w\rangle = \big(h_1, w\big), \forall w \in W_0^{1,p}(\Omega) \text{in } D'(0,T_0);$$
$$\frac{d}{dt}(v', w) + \langle \mathcal{A}v, w\rangle + \langle g(u,v)v, w\rangle = \big(h_2, w\big), \quad \forall w \in W_0^{1,p}(\Omega) \text{ in } D'(0,T_0);$$
$$u(0) = u_0, \quad u'(0) = u_1, \quad v(0) = v_0, \quad v'(0) = v_1.$$

2. Existence Results

Theorem 2.1. *Let f and g be functions of two variables such that f is continuous in the first variable and Lipschitz in the second variable and g is Lipschitz in the first and continuous in the second variable, with $f(0,0) = g(0,0) = 0$.*

$$h_1, h_2 \in L^2(0, T; L^2(\Omega)); \tag{2.1}$$
$$u_0, v_0 \in W_0^{1,p}(\Omega); \tag{2.2}$$
$$u_1, v_1 \in L^2(\Omega). \tag{2.3}$$

Then it exists $T_0 > 0$, $T_0 \in \mathbb{R}$ and functions $u : Q_{T_0} \to \mathbb{R}$ and $v : Q_{T_0} \to \mathbb{R}$ satisfying

$$u, v \in L^\infty(0, T_0; W_0^{1,p}(\Omega)); \tag{2.4}$$
$$u', v' \in L^\infty(0, T_0; L^2(\Omega)); \tag{2.5}$$
$$\frac{d}{dt}(u', w) + \langle \mathcal{A}u, w\rangle + \langle f(u,v)u, w\rangle = \big(h_1, w\big), \quad \forall w \in W_0^{1,p}(\Omega), \text{ in } D'(0,T_0); \tag{2.6}$$
$$\frac{d}{dt}(v', w) + \langle \mathcal{A}v, w\rangle + \rangle g(u,v)v, w\rangle = \big(h_2, w\big), \quad \forall w \in W_0^{1,p}(\Omega), \text{ in } D'(0,T_0); \tag{2.7}$$
$$u(0) = u_0, \quad v(0) = v_0; \tag{2.8}$$

$$u'(0) = u_1, \quad v'(0) = v_1. \tag{2.9}$$

The main tools in the proof of this theorem are the Faedo-Galerkin method and compactness arguments. Let $H_0^s(\Omega)$, with $s > m = n\big(\frac{1}{2} - \frac{1}{p}\big) + 1$ a separable Hilbert space such that $H_0^s(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$, is a continuous and dense immersion. In $H_0^s(\Omega)$, there exists a complete orthonormal hilbertian base $\{w_j\}_{j\in N}$ in $L^2(\Omega)$. We consider $V_m = [w_1, \dots, w_m]$ the subspace of $H_0^s(\Omega)$ generated by the m first vectors of the base $\{w_j\}_{j\in\mathbb{N}}$. Also, we have the following chain of continuous and dense immersions.

$$H_0^s(\Omega) \hookrightarrow W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow W^{-1,p'}(\Omega) \hookrightarrow H^{-s}(\Omega). \tag{2.10}$$

We will divide the proof in three steps: (i) Approximated Problem, (ii) A Priori Estimates I and (iii) A Priori Estimates II.

Approximated Problem. We want to find $u_m(t), v_m(t)$ in V_m satisfying the approximated problem.

$$(u_m''(t), w) + \langle \mathcal{A}u_m(t), w\rangle + \langle f(u_m(t), v_m(t))u_m(t), w\rangle = (h_1(t), w), \tag{2.11}$$

$$(v_m''(t), w) + \langle \mathcal{A}v_m(t), w\rangle + \langle g(u_m(t), v_m(t))v_m(t), w\rangle = (h_2(t), w), \tag{2.12}$$

for all $w \in V_m$; and

$$\begin{gathered} u_m(0) = u_{0m}, \quad u_m'(0) = u_{1m}, \\ v_m(0) = v_{0m}, \quad v_m'(0) = v_{1m}; \end{gathered} \tag{2.13}$$

So that

$$\begin{gathered} u_{0m} \to u_0, \quad v_{0m} \to v_0, \quad \text{in } W_0^{1,p}(\Omega); \\ u_{1m} \to u_1, \quad v_{1m} \to v_1, \quad \text{in } L^2(\Omega). \end{gathered}$$

It can be shown that the above system satisfies the Caracthodory's conditions; therefore there exists solutions $u_m(t), v_m(t)$ in $[0, t_m)$, $t_m < T$ satisfying (2.11)–(2.13).

A priori estimates I. Let us consider $w = 2u_m'(t)$ in (2.11). It follows that

$$\begin{aligned} &2(u_m''(t), u_m'(t)) + 2\langle \mathcal{A}u_m(t), u_m'(t)\rangle + 2\langle f(u_m(t), v_m(t))u_m(t), u_m'(t)\rangle \\ &= (h_1(t), u_m'(t)). \end{aligned}$$

Thus

$$\frac{d}{dt}|u_m'(t)|^2 + \frac{2}{p}\frac{d}{dt}\|u_m(t)\|_0^p = 2(h_1(t), u_m'(t)) - 2\langle f(u_m(t), v_m(t))u_m(t), u_m'(t)\rangle.$$

Similarly, setting $w = 2v_m'(t)$ in (2.12) it follows that

$$\frac{d}{dt}|v_m'(t)|^2 + \frac{2}{p}\frac{d}{dt}\|v_m(t)\|_0^p = 2(h_2(t), v_m'(t)) - 2\langle g(u_m(t), v_m(t))u_m(t), v_m'(t)\rangle.$$

Summing the two equalities above, then integrating from 0 to t, $t < t_m$, and using the Cauchy-Schwarz's inequality and $ab \le \frac{a^2+b^2}{2}$, we obtain

$$\begin{aligned} &|u_m'(t)|^2 + |v_m'(t)|^2 + \frac{2}{p}\|u_m(t)\|_0^p + \frac{2}{p}\|v_m(t)\|_0^p \\ &\le |u_m'(0)|^2 + |v_m'(0)|^2 + \frac{2}{p}\|u_m(0)\|_0^p + \frac{2}{p}\|v_m(0)\|_0^p \\ &\quad + 2\int_0^t \int_\Omega |f(u_m(s), v_m(s))||u_m(s)||u_m'(s)|ds \end{aligned}$$

$$
\begin{aligned}
&+ 2\int_0^t \int_\Omega |g(u_m(s), v_m(s))||v_m(s)||v_m'(s)|ds \\
&+ \int_0^t \big(|u_m'(s)|^2 + |v_m'(s)|^2\big)ds + \int_0^T \big(|h_1(t)|^2 + |h_2(t)|^2\big)dt.
\end{aligned}
$$

From (2.1), (2), and (2), it follows that

$$
\begin{aligned}
&|u_m'(t)|^2 + |v_m'(t)|^2 + \frac{2}{p}\|u_m(t)\|_0^p + \frac{2}{p}\|v_m(t)\|_0^p \\
&\le C + \int_0^t \big(|u_m'(s)|^2 + |v_m'(s)|^2\big)ds \\
&\quad + 2\int_0^t |f(u_m(s), v_m(s))||u_m(s)||u_m'(s)|ds \\
&\quad + 2\int_0^t |g(u_m(s), v_m(s))||v_m(s)||v_m'(s)|ds.
\end{aligned} \tag{2.14}
$$

From the Sobolev immersions it is well known that

$$
W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \forall 1 \le q \le \frac{np}{n-p}.
$$

Let $\alpha, \beta > 0$, such that $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{2} = 1$, with $1 \le \alpha, \beta \le \frac{np}{n-p}$.

Now, using Holder and Young inequalities, the inequality $ab \le \frac{a^2+b^2}{2}$ and the hypothesis over f, we have

$$
\begin{aligned}
&2\int_0^t \int_\Omega |f(u_m(s), v_m(s))||u_m(s)||u_m'(s)|ds \\
&\le C\int_0^t \int_\Omega |v_m(s)||u_m(s)||u_m'(s)|ds \\
&\le C\int_0^t \Big(\int_\Omega |v_m(s)|^\alpha\Big)^{\frac{1}{\alpha}} \Big(\int_\Omega |u_m(s)|^\beta\Big)^{\frac{1}{\beta}} \Big(\int_\Omega |u_m'(s)|^2\Big)^2 \\
&= C\int_0^t |v_m(s)|_{L^\alpha(\Omega)} |u_m(s)|_{L^\beta(\Omega)} |u_m'(s)|_{L^2(\Omega)} ds \\
&\le C\int_0^t \Big\{\frac{1}{p}|v_m(s)|^p_{L^\alpha(\Omega)} + \frac{p-1}{p}|u_m(s)|^{\frac{p}{p-1}}_{L^\beta(\Omega)}\Big\}|u_m'(s)|_{L^2(\Omega)} ds \\
&\le C\int_0^t \Big\{\frac{1}{p}|v_m(s)|^p_{L^\alpha(\Omega)} + \frac{1}{p}|u_m(s)|^{\frac{p}{p-1}(p-1)}_{L^\beta(\Omega)} + \frac{p-2}{p-1}\Big\}|u_m'(s)|_{L^2(\Omega)} ds \\
&= C\int_0^t \Big\{\frac{1}{p}|v_m(s)|^p_{L^\alpha(\Omega)} + \frac{1}{p}|u_m(s)|^p_{L^\beta(\Omega)} + \frac{p-2}{p-1}\Big\}|u_m'(s)|_{L^2(\Omega)} ds \\
&\le C\int_0^t \Big\{\frac{1}{p}|v_m(s)|^p_{L^\alpha(\Omega)} + \frac{1}{p}|u_m(s)|^p_{L^\beta(\Omega)} + \frac{p-2}{p-1}\Big\}^2 + |u_m'(s)|^2_{L^2(\Omega)} ds \\
&\le C\int_0^t \Big\{\frac{1}{p^2}|v_m(s)|^{2p}_{L^\alpha(\Omega)} + \frac{1}{p^2}|u_m(s)|^{2p}_{L^\beta(\Omega)} + (\frac{p-2}{p-1})^2 + |u_m'(s)|^2_{L^2(\Omega)}\Big\} ds \\
&\le C\int_0^t \Big\{\frac{1}{p^2}|v_m(s)|^{2p}_{L^\alpha(\Omega)} + \frac{1}{p^2}|u_m(s)|^{2p}_{L^\beta(\Omega)} + 1 + |u_m'(s)|^2_{L^2(\Omega)}\Big\} ds.
\end{aligned}
$$

Since $W_0^{1,p}(\Omega) \hookrightarrow L^\alpha(\Omega)$ and $W_0^{1,p}(\Omega) \hookrightarrow L^\beta(\Omega)$, it follows that

$$\begin{aligned} &2\int_0^t \int_\Omega |f(u_m(s), v_m(s))||u_m(s)||u_m'(s)|ds \\ &\leq C\int_0^t \Big\{\frac{1}{p^2}\|v_m(s)\|_0^{2p} + \frac{1}{p^2}\|u_m(s)\|_0^{2p} + 1 + |u_m'(s)|^2_{L^2(\Omega)}\Big\}ds. \end{aligned} \tag{2.15}$$

Similarly, we have

$$\begin{aligned} &2\int_0^t \int_\Omega |g(u_m(s), v_m(s))||v_m(s)||v_m'(s)|ds \\ &\leq C\int_0^t \Big\{\frac{1}{p^2}\|u_m(s)\|_0^{2p} + \frac{1}{p^2}\|v_m(s)\|_0^{2p} + 1 + |v_m'(s)|^2_{L^2(\Omega)}\Big\}ds. \end{aligned} \tag{2.16}$$

Substituting, (2.15) and (2.16) in (2.14),

$$\begin{aligned} &|u_m'(t)|^2 + |v_m'(t)|^2 + \frac{2}{p}\|u_m(t)\|_0^p + \frac{2}{p}\|v_m(t)\|_0^p \\ &\leq C + C\int_0^t \big(|u_m'(s)|^2 + |v_m'(s)|^2\big)ds + C\int_0^t \big\{\|u_m(s)\|_0^{2p} + \|v_m(s)\|_0^{2p}\big\} \\ &\quad + C\int_0^t 2\,ds \\ &\leq C + C\int_0^t \big(|u_m'(s)|^2 + |v_m'(s)|^2\big)ds + C\int_0^t \big\{\|u_m(s)\|_0^{2p} + \|v_m(s)\|_0^{2p}\big\} \\ &\quad + C\int_0^T 2\,ds \\ &\leq C + C\int_0^t \big(|u_m'(s)|^2 + |v_m'(s)|^2\big)ds + C\int_0^t \big\{\|u_m(s)\|_0^{2p} + \|v_m(s)\|_0^{2p}\big\}. \end{aligned} \tag{2.17}$$

Note that

$$\begin{aligned} &\frac{2}{p}|u_m'(t)|^2 + \frac{2}{p}|v_m'(t)|^2 + \frac{2}{p}\|u_m(t)\|_0^p + \frac{2}{p}\|v_m(t)\|_0^p \\ &\leq |u_m'(t)|^2 + |v_m'(t)|^2 + \frac{2}{p}\|u_m(t)\|_0^p + \frac{2}{p}\|v_m(t)\|_0^p, \end{aligned}$$

with $p > 2$, It follows that

$$\begin{aligned} &|u_m'(t)|^2 + |v_m'(t)|^2 + \|u_m(t)\|_0^p + \|v_m(t)\|_0^p \\ &\leq C + C\int_0^t \big(|u_m'(s)|^2 + |v_m'(s)|^2\big)ds + C\int_0^t \big\{\|u_m(s)\|_0^{2p} + \|v_m(s)\|_0^{2p}\big\} \\ &\leq C + C\int_0^t \Big\{\big(|u_m'(s)|^2 + |v_m'(s)|^2\big)^2 + \big(\|u_m(s)\|_0^p + \|v_m(s)\|_0^p\big)^2 \\ &\quad + 2\big(|u_m'(s)|^2 + |v_m'(s)|^2\big)\big(\|u_m(s)\|_0^p + \|v_m(s)\|_0^p\big)\Big\}ds \\ &\quad + C\int_0^t \big\{|u_m'(s)|^2 + |v_m'(s)|^2 + \|u_m(s)\|_0^p + \|v_m(s)\|_0^p\big\}ds \\ &= C + C\int_0^t \big\{|u_m'(s)|^2 + |v_m'(s)|^2 + \|u_m(s)\|_0^p + \|v_m(s)\|_0^p\big\}^2 ds \end{aligned}$$

$$+ C\int_0^t \big\{|u_m'(s)|^2 + |v_m'(s)|^2 + \|u_m(s)\|_0^p + \|v_m(s)\|_0^p\big\}ds.$$

By setting

$$\phi(t) = |u_m'(t)|^2 + |v_m'(t)|^2 + \|u_m(t)\|_0^p + \|v_m(t)\|_0^p,$$

the above inequality can be rewritten as

$$\phi(t) \le C + C\int_0^t \big\{\phi(s) + \phi^2(s)\big\}ds. \tag{2.18}$$

Then, by Lemma 1.1, there exists $T_0 \in \mathbb{R}$, with $0 < T_0 < T$, such that ϕ is bounded in $[0, T_0)$. From this, we have

$$|u_m'(t)|^2 + |v_m'(t)|^2 + \|u_m(t)\|_0^p + \|v_m(t)\|_0^p \le C \quad \forall t \in [0, T_0), \quad \forall m \in \mathbb{N}. \tag{2.19}$$

Therefore, by prolongation results, we can extend the solutions $u_m(t), v_m(t)$, to the interval $[0, T_0]$.

We will estimate, now, the second derivatives $u_m''(t)$, $v_m''(t)$. Since the procedure, to estimates $u_m''(t)$ and $v_m''(t)$ are similar, we will fix our attention only on bounding $u_m''(t)$.

2.1. **A priori Estimates II.** Let $P_m : L^2(\Omega) \to V_m \subset L^2(\Omega)$ be

$$P_m(h) = \sum_{j=1}^m (h, w_j)w_j,$$

the projection operator on $L^2(\Omega)$. Observe that $P_m = P_m^*$ and $P_m \in \mathcal{L}(H_0^s(\Omega))$. Now, by the approximate equation (2.12),

$$(u_m''(t), w) + \langle \mathcal{A}u_m(t), w\rangle + \langle f(u_m(t), v_m(t))u_m(t), w\rangle = (h_1(t), w) \tag{2.20}$$

for all $w \in V_m$. By the chain of immersions (2.10) we have

$$\langle u_m''(t) + \mathcal{A}u_m(t) + f(u_m(t), v_m(t))u_m(t) - h_1(t), w\rangle_{H^{-s}(\Omega), H_0^s(\Omega)} = 0,$$

for all $w \in V_m$. From this equality and the fact that $P_m w = w, \forall w \in V_m$, we have

$$P_m^*(u_m''(t) + \mathcal{A}u_m(t) + f(u_m(t), v_m(t))u_m(t) - h_1(t)) = 0$$

in V_m. From this, by the linearity of P_m^*, the fact that $u_m'' \in V_m$, and by the continuous and dense immersions, we have

$$u_m''(t) = -P_m^*(\mathcal{A}u_m(t)) - P_m^*(f(u_m(t), v_m(t))u_m(t)) + P_m^*(h_1(t))$$

in $H^{-s}(\Omega)$. Thus

$$\begin{aligned}\|u_m''(t)\|_{H^{-s}(\Omega)} &\le \|P_m^*(f(u_m(t), v_m(t))u_m(t))\|_{H^{-s}(\Omega)} \\ &\quad + \|P_m^*(\mathcal{A}u_m(t))\|_{H^{-s}(\Omega)} + \|P_m^*(h_1(t))\|_{H^{-s}(\Omega)}\end{aligned}$$

With $P_m \in \mathcal{L}(H_0^s(\Omega))$ which implies $P_m^* \in \mathcal{L}(H^{-s}(\Omega))$. Since $W^{-1,p'}(\Omega) \hookrightarrow H^{-s}(\Omega)$, it follows that $P_m^* \in \mathcal{L}(W^{-1,p'}(\Omega), H^{-s}(\Omega))$, Then

$$\|P_m^*(Au_m(t))\|_{H^{-s}(\Omega)} \le C\|(\mathcal{A}u_m(t))\|_{W^{-1,p'}(\Omega)} \le C\|u_m(t)\|_0^{p-1}. \tag{2.21}$$

Since, $L^2(\Omega) \hookrightarrow H^{-s}(\Omega)$, we have $P_m^* \in \mathcal{L}(L^2(\Omega), H^{-s}(\Omega)$. Furthermore,

$$\|P_m^*(h_1(t))\|_{H^{-s}(\Omega)} \le C|h_1(t)|_{L^2(\Omega)}. \tag{2.22}$$

Now, to bound the term $\|P_m^*(f(u_m(t), v_m(t))u_m(t))\|_{H^{-s}(\Omega)}$, it is necessary to place $f(u_m(t), v_m(t))u_m(t)$ in some space contained in $H^{-s}(\Omega)$. Let $\gamma, \theta \in [1, \frac{np}{n-p}]$, such that $\frac{1}{\gamma}+\frac{1}{\theta}=1$. Since $W_0^{1,p}(\Omega)\hookrightarrow L^q(\Omega)$ for $1\le q\le \frac{np}{n-p}$, we have, in particular $W_0^{1,p}(\Omega)\hookrightarrow L^\gamma(\Omega)$. Therefore,

$$\big(L^\gamma(\Omega)\big)'\hookrightarrow W^{-1,p'}(\Omega).$$

From the chain of immersions (2.10), we have $W^{-1,p'}(\Omega)\hookrightarrow H^{-s}(\Omega)$, from where

$$L^\theta(\Omega)=\big(L^\gamma(\Omega)\big)'\hookrightarrow H^{-s}(\Omega) \tag{2.23}$$

Now, it is sufficient to show that $f(u_m(t),v_m(t))u_m(t)\in L^\theta(\Omega)$. From the Hölder inequality and the hypothesis on f we have

$$\begin{aligned}\int_\Omega |f(u_m(s),v_m(s))u_m(s)|^\theta dx &= \int_\Omega |f(u_m(s),v_m(s))|^\theta |u_m(s)|^\theta dx\\ &\le C_f^\theta \int_\Omega |v_m(s))|^\theta |u_m(s)|^\theta dx\\ &\le C_f^\theta\Big(\int_\Omega |v_m(s))|^{\alpha'\theta}\Big)^{1/\alpha'}\Big(\int_\Omega |u_m(s))|^{\beta'\theta}\Big)^{\frac{1}{\beta'}},\end{aligned} \tag{2.24}$$

where C_f is the Lipschitz constant, associated f and $\frac{1}{\alpha'}+\frac{1}{\beta'}=1$.

If $\theta\alpha'\le\frac{np}{n-p}$ and $\theta\beta'\le\frac{np}{n-p}$, then

$$\theta\le\frac{1}{\alpha'}\frac{np}{(n-p)},\quad\text{and}\quad \theta\le\frac{1}{\beta'}\frac{np}{(n-p)},$$

from which,

$$2\theta\le\big(\frac{1}{\alpha'}+\frac{1}{\beta'}\big)\frac{np}{n-p}.$$

Then, we have

$$1\le\theta\le\frac{np}{2(n-p)}<\frac{np}{n-p}.$$

Noticing that $W_0^{1,p}(\Omega)\hookrightarrow L^{\theta\alpha'}(\Omega)$ and $W_0^{1,p}(\Omega)\hookrightarrow L^{\theta\beta'}(\Omega)$, we have

$$\int_\Omega |f(u_m(s),v_m(s))u_m(s)|^\theta dx\le C_f^\theta|v_m(t)|^\theta_{L^{\alpha'\theta}}|u_m(t)|^\theta_{L^{\beta'\theta}}\le C\|u_m(t)\|_0^\theta\|v_m(t)\|_0^\theta.$$

From this estimate and (2.19), it follows

$$\int_\Omega |f(u_m(s),v_m(s))u_m(s)|^\theta dx<\infty; \tag{2.25}$$

that is,

$$f(u_m(t),v_m(t))u_m(t)\in L^\theta(\Omega)=\big(L^\gamma(\Omega)\big)',\quad \text{for } 1\le\theta\le\frac{np}{2(n-p)}, \tag{2.26}$$

and

$$\|f(u_m(t),v_m(t))u_m(t)\|_{L^\theta(\Omega)}\le C,\quad \forall m,\ t\in[0,T_0] \tag{2.27}$$

Similarly, we have

$$\|g(u_m(t),v_m(t))v_m(t)\|_{L^\theta(\Omega)}\le C,\quad \forall m,\ t\in[0,T_0] \tag{2.28}$$

We will also need that $f(u_m(t),v_m(t))u_m^2(t)\in L^\theta(\Omega)$. In fact, by Hölder inequality,

$$\int_\Omega |f(u_m(s),v_m(s))u_m^2(s)|^\theta dx$$

$$
\begin{aligned}
&= \int_\Omega |f(u_m(s), v_m(s))|^\theta |u_m^2(s)|^\theta dx \\
&\le C_f^\theta \int_\Omega |v_m(s))|^\theta |u_m(s)|^\theta |u_m(s)|^\theta dx \\
&\le C_f^\theta \Big(\int_\Omega |v_m(s))|^{\xi\theta} \Big)^{\frac{1}{\xi}} \Big(\int_\Omega |u_m(s))|^{\delta\theta} \Big)^{1/\delta} \Big(\int_\Omega |u_m(s))|^{\omega\theta} \Big)^{1/\omega},
\end{aligned}
$$

where C_f is the Lipschitz constant, associated to f and $\frac{1}{\delta} + \frac{1}{\omega} + \frac{1}{\xi} = 1$. If $\theta\xi \le \frac{np}{n-p}$, $\theta\delta \le \frac{np}{n-p}$ and $\theta\omega \le \frac{np}{n-p}$ then

$$
\theta \le \frac{1}{\xi} \frac{np}{n-p}, \quad \theta \le \frac{1}{\delta} \frac{np}{n-p}, \quad \theta \le \frac{1}{\omega} \frac{np}{n-p}
$$

which implies

$$
3\theta \le \big(\frac{1}{\xi} + \frac{1}{\delta} + \frac{1}{\omega}\big) \frac{np}{n-p}.
$$

Then

$$
1 \le \theta \le \frac{np}{3(n-p)} < \frac{np}{n-p}.
$$

Observing that $W_0^{1,p}(\Omega) \hookrightarrow L^{\theta\xi}(\Omega)$, $W_0^{1,p}(\Omega) \hookrightarrow L^{\theta\delta}(\Omega)$ and $W_0^{1,p}(\Omega) \hookrightarrow L^{\theta\omega}(\Omega)$, it follows that

$$
\begin{aligned}
\int_\Omega |f(u_m(s), v_m(s)) u_m^2(s)|^\theta dx &\le C_f^\theta |v_m(t)|_{L^{\xi\theta}}^\theta |u_m(t)|_{L^{\omega\theta}}^\theta |u_m(t)|_{L^{\delta\theta}}^\theta \\
&\le C \|u_m(t)\|_0^{2\theta} \|v_m(t)\|_0^\theta .
\end{aligned}
\tag{2.29}
$$

This estimate and (2.19) lead us to

$$
\int_\Omega |f(u_m(s), v_m(s)) u_m^2(s)|^\theta dx < \infty;
$$

that is,

$$
f(u_m(t), v_m(t)) u_m^2(t) \in L^\theta(\Omega) = \big(L^\gamma(\Omega)\big)', \quad \text{for } 1 \le \theta \le \frac{np}{3(n-p)}, \tag{2.30}
$$

$$
\|f(u_m(t), v_m(t)) u_m^2(t)\|_{L^\theta(\Omega)} \le C, \quad \forall m,\ t \in [0, T_0] \tag{2.31}
$$

Similarly, we have

$$
\|g(u_m(t), v_m(t)) v_m^2(t)\|_{L^\theta(\Omega)} \le C, \quad \forall m,\ t \in [0, T_0] \tag{2.32}
$$

Note that if $\theta \le \frac{np}{3(n-p)}$, we still have (2.26) and (2.30), because $\frac{np}{3(n-p)} < \frac{np}{2(n-p)}$. Thus, as $L^\theta(\Omega) \hookrightarrow H^{-s}(\Omega)$, we have that $P_m^* \in \mathcal{L}(L^\theta(\Omega), H^{-s}(\Omega))$. Therefore

$$
\|P_m^*(f(u_m(t), v_m(t)) u_m(t))\|_{H^{-s}(\Omega)} \le C \|f(u_m(t), v_m(t)) u_m(t)\|_{L^\theta(\Omega)}. \tag{2.33}
$$

Hence, from the estimates (2.21), (2.22) and (2.33). we have

$$
\|u_m''(t)\|_{H^{-s}(\Omega)} \le C\big\{ \|u_m(t)\|_0^{p-1} + \|f(u_m(t), v_m(t)) u_m(t)\|_{L^\theta(\Omega)} + |h_1(t)| \big\}.
$$

From this inequality, it results

$$
\begin{aligned}
\int_0^{T_0} \|u_m''(t)\|_{H^{-s}(\Omega)}^2 dt \le C\big\{ &\int_0^{T_0} \|u_m(t)\|_0^{2(p-1)} dt + \int_0^{T_0} |h_1(t)|^2 dt \\
&+ \int_0^{T_0} \|f(u_m(t), v_m(t)) u_m(t)\|_{L^\theta(\Omega)}^2 dt \big\}.
\end{aligned}
$$

Therefore, from (2.17), (2.25) and (2.1), we conclude that

$$\|u''_m(t)\|_{L^2(0,T_0;H^{-s}(\Omega)} \leq C, \quad \forall m \in \mathbb{N}. \tag{2.34}$$

Arguing in a similar way, one can deduce that

$$\|v''_m(t)\|_{L^2(0,T_0;H^{-s}(\Omega)} \leq C, \forall m \in \mathbb{N}. \tag{2.35}$$

From (2.19), we have

$$\begin{gathered}\|u_m(t)\|_0 \leq C \quad \text{and} \quad \|v_m(t)\|_0 \leq C, \quad \forall m,\ t \in [0,T_0]. \\ |u'_m(t)| \leq C \quad \text{and} \quad |v'_m(t)| \leq C, \quad \forall m,\ t \in [0,T_0].\end{gathered}$$

From where, it follows that $\operatorname{ess\,sup}_{t\in[0,T_0]} \|u_m(t)\|_0 \leq C$; that is

$$\|u_m\|_{L^\infty(0,T_0;W_0^{1,p}(\Omega))} \leq C, \quad \forall m \in \mathbb{N}. \tag{2.36}$$

Similarly, we have

$$\|v_m\|_{L^\infty(0,T_0;W_0^{1,p}(\Omega))} \leq C, \quad \forall m \in \mathbb{N}; \tag{2.37}$$

$$\|u'_m\|_{L^\infty(0,T_0;L^2(\Omega))} \leq C, \quad \forall m \in \mathbb{N}; \tag{2.38}$$

$$\|v'_m\|_{L^\infty(0,T_0;L^2(\Omega))} \leq C, \quad \forall m \in \mathbb{N}; \tag{2.39}$$

Therefore, from (2.27), (2.28), (2.31), (2.32), (2.34), (2.35), (2.36), (2.37), (2.38), (2.39), we have

$$(u_m)_m, (v_m)_m \quad \text{are bounded in } L^\infty(0,T_0;W_0^{1,p}(\Omega)); \tag{2.40}$$

$$(u'_m)_m, (v'_m)_m \quad \text{are bounded in } L^\infty(0,T_0;L^2(\Omega)); \tag{2.41}$$

$$(u''_m)_m, (v''_m)_m \quad \text{are bounded in } L^2(0,T_0;H^{-s}(\Omega)); \tag{2.42}$$

$$(f(u_m,v_m)u_m)_m, (g(u_m,v_m)v_m)_m \quad \text{are bounded in } L^\infty(0,T_0;L^\theta(\Omega)); \tag{2.43}$$

$$(f(u_m,v_m)u_m^2)_m, (g(u_m,v_m)v_m^2)_m \quad \text{are bounded in } L^\infty(0,T_0;L^\theta(\Omega)); \tag{2.44}$$

Furthermore, since $\mathcal{A}$ is bounded, we have

$$(\mathcal{A}u_m)_m, (\mathcal{A}v_m)_m \quad \text{are bounded in } L^\infty(0,T_0;W^{-1,p'}(\Omega)).$$

Taking Limits. From the estimates and Banach-Alaoglu-Boubarki theorem guarantee the existence of subsequences $(u_\nu)_\nu, (v_\nu)_\nu$ of $(u_m)_m, (v_m)_m$, respectively, such that

$$u_\nu \overset{*}{\rightharpoonup} u, \quad v_\nu \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(0,T_0;W_0^{1,p}(\Omega)). \tag{2.45}$$

$$u'_\nu \overset{*}{\rightharpoonup} u', \quad v'_\nu \overset{*}{\rightharpoonup} v' \quad \text{in } L^\infty(0,T_0;L^2(\Omega)). \tag{2.46}$$

$$u''_\nu \overset{*}{\rightharpoonup} u'', \quad v''_\nu \overset{*}{\rightharpoonup} v'' \quad \text{in } L^2(0,T_0;H^{-s}(\Omega)). \tag{2.47}$$

$$Au_\nu \overset{*}{\rightharpoonup} \chi, \quad \mathcal{A}v_\nu \overset{*}{\rightharpoonup} \eta \quad \text{in } L^\infty(0,T_0;W^{-1,p'}(\Omega)). \tag{2.48}$$

As $L^2(0,T_0;H^{-s}(\Omega))$ is reflexive, the convergence (2.47) becomes

$$u''_\nu \rightharpoonup u'', v''_\nu \rightharpoonup v'' \quad \text{in } L^2(0,T_0;H^{-s}(\Omega)). \tag{2.49}$$

Let us consider the approximate equation (2.11) in the form

$$(u''_\nu(t),w) + \langle \mathcal{A}u_\nu(t),w\rangle + \langle f(u_\nu,(t),v_\nu,(t))u_\nu(t),w\rangle = (h_1(t),w) \quad \forall w \in V_m,\ \nu \geq m$$

Now, multiplying the above equality by $\varphi \in D(0,T_0)$ and integrating from 0 for T_0 we obtain

$$\int_0^{T_0}(u''_\nu(t),w)\varphi dt + \int_0^{T_0}\langle \mathcal{A}u_\nu(t),w\rangle\varphi dt + \int_0^{T_0}\langle f(u_{\nu,}(t),v_{\nu,}(t))u_\nu(t),w\rangle\varphi dt$$
$$= \int_0^{T_0}(h_1(t),w)\varphi dt \quad \forall w \in V_m,\ \nu \geq m.$$

Integrating by parts, we obtain

$$-\int_0^{T_0}(u'_\nu(t),w)\varphi' dt + \int_0^{T_0}\langle \mathcal{A}u_\nu(t),w\rangle\varphi dt + \int_0^{T_0}\langle f(u_{\nu,}(t),v_{\nu,}(t))u_\nu(t),w\rangle\varphi dt$$
$$= \int_0^{T_0}(h_1(t),w)\varphi dt \quad \forall w \in V_m,\ \nu \geq m. \tag{2.50}$$

With $u'_\nu \overset{*}{\rightharpoonup} u'$ in $L^\infty(0,T_0;L^2(\Omega)) = \big(L^1(0,T_0;L^2(\Omega))\big)'$ then

$$\langle u'_\nu,\phi\rangle \to \langle u',\phi\rangle, \quad \forall \phi \in L^1(0,T_0;L^2(\Omega)). \tag{2.51}$$

Convergence (2.51) with $\langle u'_\nu,\phi\rangle = \int_0^{T_0}(u'_\nu(t),\phi(t))dt$, and assuming $\phi(x,t) = w(x)\psi(t)$ imply hat

$$\int_0^{T_0}(u'_\nu(t),\phi(t))dt = \int_0^{T_0}(u'_\nu(t),w(x))\psi(t)dt, \forall w \in L^2(\Omega), \quad \forall \psi \in L^1(0,T_0).$$

Consequently, for all $w \in L^2(\Omega)$ and all $\psi \in L^1(0,T_0)$,

$$\int_0^{T_0}(u'_\nu(t),w(x))\psi(t)dt \to \int_0^{T_0}(u'(t),w(x))\psi(t)dt\,.$$

In fact,

$$\int_0^{T_0}(u'_\nu(t),w(x))\varphi'(t)dt \to \int_0^{T_0}(u'(t),w(x))\varphi'(t)dt,$$

for all $w \in V_m \subset W_0^{1,p}(\Omega) \subset L^2(\Omega)$ and all $\psi = \varphi'$, $\varphi \in D(0,T_0) \subset L^1(0,T_0)$. In a similar way,

$$\int_0^{T_0} < \mathcal{A}u_\nu(t),w(x) > \psi(t)dt \to \int_0^{T_0} < \chi(t),w(x) > \psi(t)dt,$$

for all $w \in W_0^{1,p}(\Omega)$ and all $\psi \in L^1(0,T_0)$. In fact,

$$\int_0^{T_0}(\mathcal{A}u_\nu(t),w(x))\varphi(t)dt \to \int_0^{T_0}(\chi(t),w(x))\varphi(t)dt,$$

for all $w \in V_m \subset W_0^{1,p}(\Omega)$ and all $\varphi \in D(0,T_0) \subset L^1(0,T_0)$.

From (2.24), we have the existence of a subsequence $(f(u_{\nu,},v_{\nu,})u_\nu)_\nu$ such that

$$f(u_{\nu,},v_{\nu,})u_\nu \overset{*}{\rightharpoonup} \lambda, \quad \text{in } L^\infty(0,T_0;L^\theta(\Omega)). \tag{2.52}$$

Since $L^\infty(0,T_0;L^\theta(\Omega)) \hookrightarrow L^\theta(0,T_0;L^\theta(\Omega))$, we have from (2.29) that

$$(f(u_m(t),v_m(t))u_m(t))_m, (g(u_m(t),v_m(t))v_m(t))_m$$

are bounded in $L^\theta(0,T_0;L^\theta(\Omega))$; Thus we guarantee the existence of a subsequence, denoted as above, such that

$$f(u_{\nu,},v_{\nu,})u_\nu \rightharpoonup \lambda, \quad \text{in } L^\theta(0,T_0;L^\theta(\Omega)). \tag{2.53}$$

Since

$$(u'_m)_m, \quad \text{is bounded in } L^\infty(0,T_0;L^2(\Omega)),$$
$$(u_m)_m, \quad \text{is bounded in } L^\infty(0,T_0;W_0^{1,p}(\Omega)) W_0^{1,p}(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega),$$

we have by Aubin-Lions theorem, the existence of a subsequence $(u_\nu)_\nu$ such that

$$u_\nu \to u, \quad \text{in} L^2(0,T_0;L^2(\Omega)) \equiv L^2(Q_{T_0}) \tag{2.54}$$
$$u_\nu \to u, \quad \text{a.e. in } Q_{T_0} \tag{2.55}$$

Since, the sequences $(v_m)_m, (v'_m)_m$ satisfy the same conditions, it follows that, there exists a subsequence $(v_\nu)_\nu$ such that

$$v_\nu \to v, \quad \text{in} L^2(0,T_0;L^2(\Omega)) \equiv L^2(Q_{T_0}) \tag{2.56}$$
$$v_\nu \to v, \quad \text{a.e, in} Q_{T_0} \tag{2.57}$$

From (2.55), (2.57), and of the hypothesis on f, g, we have

$$f(u_{\nu,}, v_{\nu,})u_\nu \to f(u,v)u, \quad \text{a.e. in } Q_{T_0}. \tag{2.58}$$
$$g(u_{\nu,}, v_{\nu,})v_\nu \to g(u,v)v, \quad \text{a.e. in } Q_{T_0}. \tag{2.59}$$

From (2.27), we have

$$\|f(u_m,v_m)u_m\|_{L^\theta(Q_{T_0})} \le C, \quad \forall m,$$

where $L^\theta(Q_{T_0}) \equiv L^\theta(0,T_0;L^\theta(\Omega))$. From this and (2.58), by means of Lion's Lemma, it follows that

$$f(u_{\nu,}, v_{\nu,})u_\nu \rightharpoonup f(u,v)u, \text{in } L^\theta(Q_{T_0}),$$

for $1 \le \theta \le \frac{np}{3(n-p)}$. Therefore, from (2.53), we have $\lambda = f(u,v)u$ and from (2.52). This implies

$$f(u_{\nu,}, v_{\nu,})u_\nu \overset{*}{\rightharpoonup} f(u,v)u, \quad \text{in } L^\infty(0,T_0;L^\theta(\Omega)). \tag{2.60}$$

Similarly,

$$g(u_{\nu,}, v_{\nu,})v_\nu \overset{*}{\rightharpoonup} g(u,v)v, \quad \text{in } L^\infty(0,T_0;L^\theta(\Omega)).$$

The convergence in (2.60) implies

$$\int_0^{T_0} \big\langle f(u_\nu(t),v_\nu(t))u_\nu(t), w(x)\big\rangle \psi(t)dt \to \int_0^{T_0} \big\langle f(u(t),v(t))u(t), w(x)\big\rangle \psi(t)dt,$$

for all $w \in W_0^{1,p}(\Omega) \subset L^\gamma(\Omega)$, for all $\psi \in L^1(0,T_0)$. In fact,

$$\int_0^{T_0} \big\langle f(u_\nu(t),v_\nu(t))u_\nu(t), w(x)\big\rangle \varphi(t)dt \to \int_0^{T_0} \big\langle f(u(t),v(t))u(t), w(x)\big\rangle \varphi(t)dt,$$

for all $w \in V_m \subset W_0^{1,p}(\Omega) \subset L^\gamma(\Omega)$, for all $\varphi \in D(0,T_0) \subset L^1(0,T_0)$. Taking the limit, as $\nu \to \infty$, in (2.50) and using the convergences obtained above, we have

$$\begin{aligned} &-\int_0^{T_0} (u'(t),w)\varphi' dt + \int_0^{T_0} \langle \chi(t), w\rangle \varphi dt + \int_0^{T_0} \langle f(u(t),v(t))u(t), w\rangle \varphi dt \\ &= \int_0^{T_0} (h_1(t),w)\varphi dt, \quad \forall w \in V_m,\ \varphi \in D(0,T_0). \end{aligned} \tag{2.61}$$

Note that, with a similar reasoning for the approximate equation (2.12) we obtain

$$\begin{aligned} &-\int_0^{T_0}(v'(t),w)\varphi' dt+\int_0^{T_0}\langle\eta(t),w\rangle\varphi dt+\int_0^{T_0}\langle g(u(t),v(t))v(t),w\rangle\varphi dt \\ &=\int_0^{T_0}(h_2(t),w)\varphi dt,\quad \forall w\in V_m,\ \varphi\in D(0,T_0). \end{aligned}\tag{2.62}$$

Now, using the basis definition and the fact that V_m is dense in $W_0^{1,p}(\Omega)$, expressions (2.61) and (2.62) take the form

$$\begin{aligned} &-\int_0^{T_0}(u'(t),w)\varphi' dt+\int_0^{T_0}<\chi(t),w>\varphi dt+\int_0^{T_0}\langle f(u(t),v(t))u(t),w\rangle\varphi dt \\ &=\int_0^{T_0}(h_1(t),w)\varphi dt,\quad \forall w\in W_0^{1,p}(\Omega),\ \varphi\in D(0,T_0), \end{aligned}\tag{2.63}$$

and

$$\begin{aligned} &-\int_0^{T_0}(v'(t),w)\varphi' dt+\int_0^{T_0}\langle\eta(t),w\rangle\varphi dt+\int_0^{T_0}\langle g(u(t),v(t))v(t),w\langle\varphi dt \\ &=\int_0^{T_0}(h_2(t),w)\varphi dt,\quad \forall w\in W_0^{1,p}(\Omega),\ \varphi\in D(0,T_0). \end{aligned}\tag{2.64}$$

Note that, the mappings $t\mapsto(u'(t),w), t\mapsto(v'(t),w)$ being functions in $L^\infty(0,T_0)$, they define distributions on $(0,T_0)$. Therefore, the first integrals of (2.63), (2.64) are the derivative of these distributions. Thus, from (2.63) we have

$$\int_0^{T_0}\big\{\frac{d}{dt}(u'(t),w)+\langle\chi(t),w\rangle+\langle f(u(t),v(t))u(t),w\rangle-(h_1(t),w)\big\}\varphi dt=0$$

for all $w\in W_0^{1,p}(\Omega)$ and all $\varphi\in D(0,T_0)$. Thus,

$$\frac{d}{dt}(u'(t),w)+\langle\chi(t),w\rangle+\langle f(u(t),v(t))u(t),w\rangle=(h_1(t),w),$$

for all $w\in W_0^{1,p}(\Omega)$, in $D'(0,T_0)$. Similarly,

$$\frac{d}{dt}(v'(t),w)+\langle\eta(t),w\rangle+\langle g(u(t),v(t))v(t),w\rangle=(h_2(t),w),$$

for all $w\in W_0^{1,p}(\Omega)$, in $D'(0,T_0)$.

If one shows that $\mathcal{A}u(t)=\chi(t)$ and $\mathcal{A}v(t)=\eta(t)$, the proof of the theorem will be complete; since the verification of the initial conditions can be done in a standard way.

The monotonocity of $\mathcal{A}$ implies that

$$\int_0^{T_0}\langle\mathcal{A}u_\nu(t)-\mathcal{A}w,u_\nu-w\rangle dt\ge 0,\quad \forall w\in W_0^{1,p}(\Omega);$$

that is,

$$0\le\int_0^{T_0}\langle\mathcal{A}u_\nu(t),u_\nu\rangle dt-\int_0^{T_0}\langle\mathcal{A}u_\nu(t),w\rangle dt-\int_0^{T_0}\langle\mathcal{A}w,u_\nu(t)-w\rangle dt,$$

for all $w\in W_0^{1,p}(\Omega)$.

$$0\le\lim sup\int_0^{T_0}\langle\mathcal{A}u_\nu(t),u_\nu\rangle dt-\int_0^{T_0}\langle\chi(t),w\rangle dt-\int_0^{T_0}\langle\mathcal{A}w,u(t)-w\rangle dt,$$

for all $w \in W_0^{1,p}(\Omega)$. Considering the approximate equation (2.11) with $m = \nu$ and $w = u_\nu(t)$ we have

$$(u_\nu''(t), u_\nu(t)) + \langle \mathcal{A}u_\nu(t), u_\nu(t)\rangle + \langle f(u_\nu, v_\nu)u_\nu, u_\nu\rangle = (h_1(t), u_\nu(t)).$$

Therefore,

$$\frac{d}{dt}(u_\nu'(t), u_\nu(t)) - |u_\nu'(t)|^2 + \langle \mathcal{A}u_\nu(t), u_\nu(t)\rangle + \langle f(u_\nu, v_\nu)u_\nu, u_\nu\rangle = (h_1(t), u_\nu)$$

Integrating from 0 the T_0 we have

$$\begin{aligned} \int_0^{T_0} \langle \mathcal{A}u_\nu(t), u_\nu(t)\rangle dt = (u_\nu'(0), u_\nu(0)) - (u_\nu'(T_0), u_\nu(T_0)) + \int_0^{T_0} |u_\nu'(t)|^2 dt \\ - \int_0^{T_0} \langle f(u_\nu, v_\nu)u_\nu, u_\nu\rangle dt + \int_0^{T_0} (h_1(t), u_\nu) dt \end{aligned} \tag{2.65}$$

Recall that $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$. Since $u_\nu(0) \rightharpoonup u(0)$ in $W_0^{1,p}(\Omega)$ it implies $u_\nu(0) \to u(0) in L^2(\Omega)$. Since $u_\nu'(0) \rightharpoonup u'(0)$ in $L^2(\Omega)$, it implies

$$(u_\nu'(0), u_\nu(0)) \to (u'(0), u(0)) \quad \text{in } \mathbb{R} \tag{2.66}$$

Recall that $(u_m(T_0))_m$ is bounded in $W_0^{1,p}(\Omega)$ and $(u_m'(T_0))_m$ is bounded in $L^2(\Omega)$. Thus, there exists subsequences $(u_\nu(T_0))_\nu$ and $(u_\nu'(T_0))_\nu$ such that

$$u_\nu(T_0) \rightharpoonup u(T_0) \quad \text{in } W_0^{1,p}(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega),$$

which implies

$$\begin{gathered} u_\nu(T_0) \to u(T_0), in L^2(\Omega), \\ u_\nu'(T_0) \rightharpoonup u'(T_0) in L^2(\Omega) \end{gathered}$$

Consequently,

$$(u_\nu'(0), u_\nu(T_0)) \to (u'(T_0), u(T_0)) \quad \text{in } \mathbb{R}. \tag{2.67}$$

We have that (u_m') bounded in $L^\infty(0, T_0; L^2(\Omega))$. Since

$$L^\infty(0, T_0; L^2(\Omega)) \hookrightarrow L^2(0, T_0; L^2(\Omega)),$$

it follows that (u_m') is bounded in $L^2(0, T_0; L^2(\Omega))$. We also have that (u_m'') is bounded in $L^2(0, T_0; H^{-s}(\Omega))$. Therefore, by the Aubin-Lions Theorem, there exists a subsequence (u_ν') such that

$$u_\nu' \to u' \quad \text{in } L^2(0, T_0; L^2(\Omega)) \equiv L^2(Q_{T_0}).$$

Hence

$$\int_0^{T_0} |u_\nu'(t)|^2 dt \to \int_0^{T_0} |u'(t)|^2 dt \tag{2.68}$$

Note that

$$\langle f(u_m(t), v_m(t))u_m(t), u_m(t)\rangle_{L^\theta, L^\gamma} = \langle f(u_m(t), v_m(t))u_m^2(t), 1\rangle_{L^\theta, L^\gamma}.$$

From (2.68) we have $u_\nu^2 \to u^2$ a.e. in Q_{T_0}. Similarly

$$\int_0^{T_0} |v_\nu'(t)|^2 dt \to \int_0^{T_0} |v'(t)|^2 dt$$

hence, we have $v_\nu^2 \to v^2$ a.e. in Q_{T_0}, From (2.31), we have

$$\|f(u_\nu, v_\nu)u_\nu^2\|_{L^\theta(0, T_0; L^\theta(\Omega)) \equiv L^\theta(Q_{T_0})} \le C, \quad \forall m. \tag{2.69}$$

From this inequality and (2.44), we guarantee the existence of a subsequence such that

$$f(u_\nu, v_\nu)u_\nu^2 \overset{*}{\rightharpoonup} \sigma \quad \text{in } L^\infty(0,T_0;L^\theta(\Omega)) \tag{2.70}$$

$$f(u_\nu, v_\nu)u_\nu^2 \rightharpoonup \sigma \quad \text{in } L^\theta(0,T_0;L^\theta(\Omega)) \tag{2.71}$$

Thus, from (2.55), (2.57) and the hypotheses on f, g, we have that

$$f(u_\nu, v_\nu)u_\nu^2 \to f(u,v)u^2 \quad \text{a.e. in } Q_{T_0}, \tag{2.72}$$

$$g(u_\nu, v_\nu)u_\nu^2 \to g(u,v)u^2 \quad \text{a.e in } Q_{T_0} \tag{2.73}$$

From (2.69), (2.72) and the Lions' Lemma it follows that

$$f(u_\nu, v_\nu)u_\nu^2 \rightharpoonup f(u,v)u^2 in L^\theta(Q_{T_0}) \equiv L^\theta(0,T_0;L^\theta(\Omega)), \quad \text{for } 1 \le \theta \le \frac{np}{3(n-p)}$$

From this convergence and (2.71), we have $\sigma = f(u,v)u^2$ and from (2.70),

$$f(u_\nu, v_\nu)u_\nu^2 \overset{*}{\rightharpoonup} f(u,v)u^2 \quad \text{in } L^\infty(0,T_0;L^\theta(\Omega)). \tag{2.74}$$

Similarly,

$$g(u_\nu, v_\nu)v_\nu^2 \overset{*}{\rightharpoonup} g(u,v)u^2 in L^\infty(0,T_0;L^\theta(\Omega)).$$

The convergence (2.74) implies

$$\langle f(u_\nu, v_\nu)u_\nu^2, \psi\rangle \to \langle f(u,v)u^2, \psi\rangle, \quad \forall \psi \in L^1(0,T_0;L^\gamma(\Omega))$$

or better

$$\int_0^{T_0} \langle f(u_\nu, v_\nu)u_\nu^2, w(x)\rangle \varphi(t)dt \to \int_0^{T_0} \langle f(u,v)u^2, w(x)\rangle \varphi(t)dt,$$

for all $w \in L^\gamma(\Omega)$ and all $\varphi \in L^1(0,T_0)$. When fixing $w \equiv 1$ and $\varphi \equiv 1$, we have

$$\int_0^{T_0} \langle f(u_\nu(t), v_\nu(t))u_\nu(t), u_\nu(t)\rangle dt = \int_0^{T_0} \langle f(u_\nu(t), v_\nu(t))u_\nu^2(t), 1\rangle dt$$

which approaches

$$\int_0^{T_0} \langle f(u(t), v(t))u^2(t), 1\rangle dt = \int_0^{T_0} \langle f(u(t), v(t))u(t), u(t)\rangle dt.$$

hence

$$\int_0^{T_0} \langle f(u_\nu(t), v_\nu(t))u_\nu(t), u_\nu(t)\rangle dt \to \int_0^{T_0} \langle f(u(t), v(t))u(t), u(t)\rangle dt, \tag{2.75}$$

as $\nu \to \infty$. Therefore, taking the limit in (2.65), using the convergence (2.66), (2.67), (2.68) and (2.75), as $\nu \to +\infty$, we have

$$\begin{aligned}\limsup \int_0^{T_0} \langle Au_\nu(t), u_\nu(t)\rangle dt &= (u'(0), u(0)) - (u'(T_0), u(T_0)) + \int_0^{T_0} |u'(t)|^2 dt \\ &\quad - \int_0^{T_0} \langle f(u(t), v(t))u(t), u(t)\rangle dt + \int_0^{T_0} (h_1(t), u(t))dt\end{aligned}$$

From this equality and (2.75), we have

$$\begin{aligned}0 \le (u'(0), u(0)) - (u'(T_0) - u(T_0)) + \int_0^{T_0} |u'(t)^2| dt - \int_0^{T_0} \langle f(u,v)u, u\rangle dt \\ - \int_0^{T_0} \langle \chi(t), w\rangle dt - \int_0^{T_0} \langle Aw, u(t) - w\rangle dt + \int_0^{T_0} (h_1(t), u(t))dt,\end{aligned} \tag{2.76}$$

for all $w \in W_0^{1,p}(\Omega)$. From the approximate equation (2.11), we have

$$(u_\nu''(t), w) + \langle Au_\nu(t), w\rangle + \langle f(u_\nu(t), v_\nu(t))u_\nu(t), w\rangle = (h_1(t), w), \quad \forall w \in V_m, \nu \geq m.$$

Now, let $\varphi \in C^1([0, T_0])$. Then

$$\begin{aligned}
&\int_0^{T_0} (u_\nu''(t), w)\varphi + \int_0^{T_0} \langle Au_\nu(t), w\rangle\varphi + \int_0^{T_0} \langle f(u_\nu(t), v_\nu(t))u_\nu(t), w\rangle\varphi \\
&= \int_0^{T_0} (h_1(t), w),
\end{aligned}$$

for all $w \in V_m$ and all $\nu \geq m$. Setting

$$\begin{aligned}
&(u_\nu'(t), w)\varphi(T_0) - (u_\nu'(0), w)\varphi(0) - \int_0^{T_0} (u_\nu'(t), w)\varphi' dt \\
&+ \int_0^{T_0} \langle Au_\nu(t), w\rangle\varphi dt + \int_0^{T_0} \langle f(u_\nu(t), v_\nu(t))u_\nu(t), w\rangle\varphi(t) dt \\
&= \int_0^{T_0} (h_1(t), w)\varphi(t) dt, \quad \forall w \in V_m,\ \varphi \in C^1([0, T_0]),\ \nu \geq m.
\end{aligned}$$

Taking into account the previous convergence statements, it follows that

$$\begin{aligned}
&(u'(T_0), w)\varphi(T_0) - (u'(0), w)\varphi(0) - \int_0^{T_0} (u'(t), w)\varphi' dt \\
&+ \int_0^{T_0} \langle \chi(t), w\rangle\varphi dt + \int_0^{T_0} \langle f(u(t), v(t))u(t), w\rangle\varphi(t) dt \\
&= \int_0^{T_0} (h_1(t), w)\varphi(t) dt, \quad \forall w \in V_m,\ \varphi \in C^1([0, T_0])
\end{aligned}$$

Using a basis argument and the fact that V_m is dense in $W_0^{1,p}(\Omega)$, it follows that

$$\begin{aligned}
&(u'(T_0), w)\varphi(T_0) - (u'(0), w)\varphi(0) - \int_0^{T_0} (u'(t), w)\varphi' dt \\
&+ \int_0^{T_0} \langle \chi(t), w\rangle\varphi dt + \int_0^{T_0} \langle f(u(t), v(t))u(t), w\rangle\varphi(t) dt \\
&= \int_0^{T_0} (h_1(t), w)\varphi(t) dt, \quad \forall w \in W_0^{1,p}(\Omega),\ \varphi \in C^1([0, T_0]).
\end{aligned} \tag{2.77}$$

Observing that the set of the linear combinations of the type $w\varphi$, with $w \in W_0^{1,p}(\Omega)$ and $\varphi \in C^1([0, T_0])$, is dense in the space

$$V = \{v \in L^2(0, T_0; W_0^{1,p}(\Omega)), v' \in L^2(0, T_0; L^2(\Omega))\}.$$

It follows that (2.77) is true in the space V.

Using the fact that,

$$\begin{gathered}
u \in L^\infty(0, T_0; W_0^{1,p}(\Omega)) \hookrightarrow L^2(0, T_0; W_0^{1,p}(\Omega)), \\
u' \in L^\infty(0, T_0; L^2(\Omega)) \hookrightarrow L^2(0, T_0; L^2(\Omega)),
\end{gathered}$$

we obtain that $u \in V$. So (2.77) takes the form

$$\begin{aligned}
&(u'(T_0), w)\varphi(T_0) - (u'(0), w)\varphi(0) \\
&- \int_0^{T_0} (u'(t), u'(t)) dt + \int_0^{T_0} \langle \chi(t), u(t)\rangle dt + \int_0^{T_0} \langle f(u, v)u, u\rangle dt
\end{aligned}$$

$$= \int_0^{T_0} (h_1(t), u(t)dt$$

Substituting this expression in (2.76), it follows that

$$0 \le \int_0^{T_0} \langle \chi(t), u(t) - w\rangle dt - \int_0^{T_0} \langle \mathcal{A}w, u(t) - w\rangle dt, \quad \forall w \in W_0^{1,p}(\Omega).$$

Let us take $w = u(t) + \lambda v(t), \lambda > 0$. Thus

$$0 \le -\int_0^{T_0} \langle \chi(t), \lambda v(t)\rangle dt + \int_0^{T_0} \langle \mathcal{A}u(t) + \lambda v(t), \lambda v(t)\rangle dt, \forall w \in W_0^{1,p}(\Omega)$$

which implies

$$0 \le -\int_0^{T_0} \langle \chi(t), \lambda v(t)\rangle dt + \int_0^{T_0} \langle \mathcal{A}(u(t) + \lambda v(t)), \lambda v(t)\rangle dt.$$

Dividing the previous inequality by λ and letting $\lambda \to 0^+$, by the hemicontinuity of $\mathcal{A}$, we have

$$0 \le -\int_0^{T_0} \langle \chi(t), v(t)\rangle dt + \int_0^{T_0} \langle \mathcal{A}(u(t)), v(t)\rangle dt, \quad \forall v \in W_0^{1,p}(\Omega).$$

Hence

$$0 \le \int_0^{T_0} \langle \mathcal{A}u(t) - \chi(t), v(t)\rangle dt, \quad \forall v \in W_0^{1,p}(\Omega).$$

Now, for $\lambda < 0$ it follows that

$$\int_0^{T_0} \langle \mathcal{A}u(t) - \chi(t), v(t)\rangle dt \le 0, \quad \forall v \in W_0^{1,p}(\Omega).$$

Therefore,

$$0 \le \int_0^{T_0} \langle \mathcal{A}u(t) - \chi(t), v(t)\rangle dt \le 0, \quad \forall v \in W_0^{1,p}(\Omega).$$

Thus $\mathcal{A}u(t) = \chi(t)$. Similarly, $\mathcal{A}v(t) = \eta(t)$. This completes the proof of the theorem.

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