Electronic Journal of Differential Equations, Vol. 2005(2005), No. 81, pp. 1–17.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

SCHOUTEN TENSOR EQUATIONS IN CONFORMAL GEOMETRY WITH PRESCRIBED BOUNDARY METRIC

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Abstract. We deform the metric conformally on a manifold with boundary. This induces a deformation of the Schouten tensor. We fix the metric at the boundary and realize a prescribed value for the product of the eigenvalues of the Schouten tensor in the interior, provided that there exists a subsolution. This problem reduces to a Monge-Ampère equation with gradient terms. The main issue is to obtain a priori estimates for the second derivatives near the boundary.

1. Introduction

Let (M^n, g_{ij}) be an n-dimensional Riemannian manifold, $n \geq 3$. The Schouten tensor (S_{ij}) of (M^n, g_{ij}) is defined as

$$S_{ij} = \tfrac{1}{n-2}\big(R_{ij} - \tfrac{1}{2(n-1)} R g_{ij}\big),$$

where (R_{ij}) and R denote the Ricci and scalar curvature of (M^n, g_{ij}), respectively. Consider the manifold $(\tilde{M}^n, \tilde{g}_{ij}) = (M^n, e^{-2u} g_{ij})$, where we have used $u \in C^2(M^n)$ to deform the metric conformally. The Schouten tensors S_{ij} of g_{ij} and $\tilde{S}_{ij}$ of $\tilde{g}_{ij}$ are related by

$$\tilde{S}_{ij} = u_{ij} + u_i u_j - \tfrac{1}{2}|\nabla u|^2 g_{ij} + S_{ij},$$

where indices of u denote covariant derivatives with respect to the background metric g_{ij}, moreover $|\nabla u|^2 = g^{ij} u_i u_j$ and $(g^{ij}) = (g_{ij})^{-1}$. Eigenvalues of the Schouten tensor are computed with respect to the background metric g_{ij}, so the product of the eigenvalues of the Schouten tensor $(\tilde{S}_{ij})$ equals a given function $s : M^n \to \mathbb{R}$, if

$$\frac{\det(u_{ij} + u_i u_j - \frac{1}{2}|\nabla u|^2 g_{ij} + S_{ij})}{e^{-2nu}\det(g_{ij})} = s(x). \tag{1.1}$$

We say that u is an admissible solution for (1.1), if the tensor in the determinant in the numerator is positive definite. At admissible solutions, (1.1) becomes an elliptic equation. As we are only interested in admissible solutions, we will always assume that s is positive.

2000 *Mathematics Subject Classification.* 53A30; 35J25; 58J32.
Key words and phrases. Schouten tensor; fully nonlinear equation; conformal geometry; Dirichlet boundary value problem.

Submitted March 15, 2004. Published July 15, 2005.

Let now M^n be compact with boundary and $\underline{u} : M^n \to \mathbb{R}$ be a smooth (up to the boundary) admissible subsolution to (1.1)

$$\frac{\det(\underline{u}_{ij} + \underline{u}_i\underline{u}_j - \frac{1}{2}|\nabla \underline{u}|^2 g_{ij} + S_{ij})}{e^{-2n\underline{u}}\det(g_{ij})} \geq s(x). \tag{1.2}$$

Assume that there exists a supersolution $\overline{u}$ to (1.1) fulfilling some technical conditions specified in Definition 2.1. Assume furthermore that M^n admits a strictly convex function χ. Without loss of generality, we have $\chi_{ij} \geq g_{ij}$ for the second covariant derivatives of χ in the matrix sense.

The conditions of the preceding paragraph are automatically fulfilled if M^n is a compact subset of flat $\mathbb{R}^n$ and $\underline{u}$ fulfills (1.2) and in addition $\det(\underline{u}_{ij}) \geq s(x)e^{-2n\underline{u}}\det(g_{ij})$ with $\underline{u}_{ij} > 0$ in the matrix sense. Then Lemma 2.2 implies the existence of a supersolution and we may take $\chi = |x|^2$.

We impose the boundary condition that the metric $\tilde{g}_{ij}$ at the boundary is prescribed,

$$\tilde{g}_{ij} = e^{-2\underline{u}}g_{ij} \quad \text{on } \partial M^n.$$

Assume that all data are smooth up to the boundary. We prove the following

Theorem 1.1. *Let M^n, g_{ij}, $\underline{u}$, $\overline{u}$, χ, and s be as above. Then there exists a metric $\tilde{g}_{ij}$, conformally equivalent to g_{ij}, with $\tilde{g}_{ij} = e^{-2\underline{u}}g_{ij}$ on ∂M^n such that the product of the eigenvalues of the Schouten tensor induced by $\tilde{g}_{ij}$ equals s.*

This follows readily from the next statement.

Theorem 1.2. *Under the assumptions stated above, there exists an admissible function $u \in C^0(M^n) \cap C^\infty(M^n \setminus \partial M^n)$ solving (1.1) such that $u = \underline{u}$ on ∂M^n.*

Recently, in a series of papers, Jeff Viaclovsky studied conformal deformations of metrics on closed manifolds and elementary symmetric functions S_k, $1 \leq k \leq n$, of the eigenvalues of the associated Schouten tensor, see e. g. [41] for existence results. Pengfei Guan, Jeff Viaclovsky, and Guofang Wang provide an estimate that can be used to show compactness of manifolds with lower bounds on elementary symmetric functions of the eigenvalues of the Schouten tensor [14]. An equation similar to the Schouten tensor equation arises in geometric optics [18, 42]. Xu-Jia Wang proved the existence of solutions to Dirichlet boundary value problems for such an equation, similar to (1.1), provided that the domains are small. In [39] we provide a transformation that shows the similarity between reflector and Schouten tensor equations. For Schouten tensor equations, Dirichlet and Neumann boundary conditions seem to be geometrically meaningful. For reflector problems, solutions fulfilling a so-called second boundary value condition describe the illumination of domains. Pengfei Guan and Xu-Jia Wang obtained local second derivative estimates [18]. This was extended by Pengfei Guan and Guofang Wang to local first and second derivative estimates in the case of elementary symmetric functions S_k of the Schouten tensor of a conformally deformed metric [16]. We will use the following special case of it

Theorem 1.3 (Pengfei Guan and Xu-Jia Wang/Pengfei Guan and Guofang Wang)**.** *Suppose f is a smooth function on $M^n \times \mathbb{R}$. Let $u \in C^4$ be an admissible solution of*

$$\log\det(u_{ij} + u_iu_j - \tfrac{1}{2}|\nabla u|^2 g_{ij} + S_{ij}) = f(x,\, u)$$

in B_r, the geodesic ball of radius r in a Riemannian manifold (M^n, g_{ij}). Then, there exists a constant $c = c(\|u\|_{C^0}, f, S_{ij}, r, M^n)$, such that

$$\|u\|_{C^2(B_{r/2})} \le c.$$

Boundary-value problems for Monge-Ampère equations have been studied by Luis Caffarelli, Louis Nirenberg, and Joel Spruck in [4] an many other people later on. For us, those articles using subsolutions as used by Bo Guan and Joel Spruck will be especially useful [12, 13, 37, 38].

There are many papers addressing Schouten tensor equations on compact manifolds, see e. g. [3, 5, 6, 14, 15, 16, 17, 19, 20, 21, 22, 23, 25, 26, 27, 28, 29, 30, 31, 33, 34, 36, 41]. There, the authors consider topological and geometrical obstructions to solutions, the space of solutions, Liouville properties, Harnack inequalities, Moser-Trudinger inequalities, existence questions, local estimates, local behavior, blow-up of solutions, and parabolic and variational approaches. If we consider the sum of the eigenvalues of the Schouten tensor, we get the Yamabe equation. The Yamabe problem has been studied on manifolds with boundary, see e. g. [1, 2, 7, 24, 35], and in many more papers on closed manifolds. The Yamabe problem gives rise to a quasilinear equation. For a fully nonlinear equation, we have to apply different methods.

The present paper addresses analytic aspects that arise in the proof of a priori estimates for an existence theorem. This combines methods for Schouten tensor equations, e. g. [16, 41], with methods for curvature equations with Dirichlet boundary conditions, e. g. [4, 12].

We can also solve Equation (1.1) on a non-compact manifold (M^n, g_{ij}).

Corollary 1.4. *Assume that there are a sequence of smooth bounded domains Ω_k, $k \in \mathbb{N}$, exhausting a non-compact manifold M^n, and functions $\underline{u}$, $\overline{u}$, s, and χ, that fulfill the conditions of Theorem 1.2 on each Ω_k instead of M^n. Then there exists an admissible function $u \in C^\infty(M^n)$ solving* (1.1).

Proof. Theorem 1.2 implies that equation (1.1) has a solution u_k on every Ω_k fulfilling the boundary condition $u = \underline{u}$ on $\partial\Omega_k$. In Ω_k, we have $\underline{u} \le u_k \le \overline{u}$, so Theorem 1.3 implies locally uniform C^2-estimates on u_k on any domain $\Omega \subset M^n$ for $k > k_0$, if $\Omega \Subset \Omega_{k_0}$. The estimates of Krylov, Safonov, Evans, and Schauder imply higher order estimates on compact subsets of M^n. Arzelà-Ascoli yields a subsequence that converges to a solution. □

Note that either $s(x)$ is not bounded below by a positive constant or the manifold with metric $e^{-2u} g_{ij}$ is non-complete. Otherwise, [14] implies a positive lower bound on the Ricci tensor, i. e. $\tilde{R}_{ij} \ge \frac{1}{c}\tilde{g}_{ij}$ for some positive constant c. This yields compactness of the manifold [11].

It is a further issue to solve similar problems for other elementary symmetric functions of the Schouten tensor. As the induced mean curvature of ∂M^n is related to the Neumann boundary condition, this is another natural boundary condition.

To show existence for a boundary value problem for fully nonlinear equations like Equation (1.1), one usually proves C^2-estimates up to the boundary. Then standard results imply C^k-bounds for $k \in \mathbb{N}$ and existence results. In our situation, however, we don't expect that C^2-estimates up to the boundary can be proved. This is due to the gradient terms appearing in the determinant in (1.1). It is possible to overcome these difficulties by considering only small domains [42]. Our method is different. We regularize the equation and prove full regularity up to the boundary for the regularized equation. Then we use the fact, that local interior C^k-estimates (Theorem 1.3) can be obtained independently of the regularization. Moreover, we can prove uniform C^1-estimates. Thus we can pass to a limit and get a solution in $C^0(M^n)\cap C^\infty(M^n\setminus\partial M^n)$.

To be more precise, we rewrite (1.1) in the form

$$\log\det(u_{ij}+u_iu_j-\tfrac{1}{2}|\nabla u|^2g_{ij}+S_{ij})=f(x,u), \tag{1.3}$$

where $f\in C^\infty(M^n\times\mathbb{R})$. Our method can actually be applied to any equation of that form provided that we have sub- and supersolutions. Thus we consider in the following equations of the form (1.3). Equation (1.3) makes sense in any dimension provided that we replace S_{ij} by a smooth tensor. In this case Theorem 1.2 is valid in any dimension. Note that even without the factor $\frac{1}{n-2}$ in the definition of the Schouten tensor, our equation is not elliptic for $n=2$ for any function u as the trace $g^{ij}(R_{ij}-\frac{1}{2}Rg_{ij})$ equals zero, so there has to be a non-positive eigenvalue of that tensor. Let $\psi:M^n\to[0,1]$ be smooth, $\psi=0$ in a neighborhood of the boundary. Then our strategy is as follows. We consider a sequence ψ_k of those functions that fulfill $\psi_k(x)=1$ for $\operatorname{dist}(x,\partial M^n)>\frac{2}{k}$, $k\in\mathbb{N}$, and boundary value problems

$$\begin{gathered}\log\det(u_{ij}+\psi u_iu_j-\tfrac{1}{2}\psi|\nabla u|^2g_{ij}+T_{ij})=f(x,u)\quad\text{in} M^n,\\ u=\underline{u}\quad\text{on }\partial M^n.\end{gathered} \tag{1.4}$$

We dropped the index k to keep the notation simple. The tensor T_{ij} coincides with S_{ij} on $\left\{x\in M^n:\operatorname{dist}(x,\partial M^n)>\frac{2}{k}\right\}$ and interpolates smoothly to S_{ij} plus a sufficiently large constant multiple of the background metric g_{ij} near the boundary. For the precise definitions, we refer to Section 2.

Our sub- and supersolutions act as barriers and imply uniform C^0-estimates. We prove uniform C^1-estimates based on the admissibility of solutions. Admissibility means here that $u_{ij}+\psi u_iu_j-\frac{1}{2}\psi|\nabla u|^2+T_{ij}$ is positive definite for those solutions. As mentioned above, we can't prove uniform C^2-estimates for u, but we get C^2-estimates that depend on ψ. These estimates guarantee, that we can apply standard methods (Evans-Krylov-Safonov theory, Schauder estimates for higher derivatives, and mapping degree theory for existence, see e. g. [10, 12, 32, 40]) to prove existence of a smooth admissible solution to (1.4). Then we use Theorem 1.3 to get uniform interior a priori estimates on compact subdomains of M^n as $\psi=1$ in a neighborhood of these subdomains for all but a finite number of regularizations. These a priori estimates suffice to pass to a subsequence and to obtain an admissible solution to (1.3) in $M^n\setminus\partial M^n$. As $u^k=u=\underline{u}$ for all solutions u^k of the regularized equation and those solutions have uniformly bounded gradients, the boundary condition is preserved when we pass to the limit and we obtain Theorem 1.2 provided that we can prove $\|u^k\|_{C^1(M^n)}\le c$ uniformly and $\|u^k\|_{C^2(M^n)}\le c(\psi)$. These estimates are proved in Lemmata 4.1 and 5.4, the crux of this paper.

Proof of Theorem 1.2. For admissible smooth solutions to (1.4), the results of Section 3 imply uniform C^0-estimates and Section 4 gives uniform C^1-estimates. The C^2-estimates proved in Section 5 depend on the regularization. The logarithm of the determinant is a strictly concave function on positive definite matrices, so the results of Krylov, Safonov, Evans, [40, 14.13/14], and Schauder estimates yield C^l-estimates on M^n, $l\in\mathbb{N}$, depending on the regularization.

Once these a priori estimates are established, existence of a solution u^k for the regularized problem (1.4) follows as in [12, Section 2.2].

On a fixed bounded subdomain $\Omega_\varepsilon := \{x : \operatorname{dist}(x, \partial M^n) \ge \varepsilon\}$, $\varepsilon > 0$, however, Theorem 1.3 implies uniform C^2-estimates for all $k \ge k_0 = k_0(\varepsilon)$. The estimates of Krylov, Safonov, Evans, and Schauder yield uniform C^l-estimates on $\Omega_{2\varepsilon}$, $l \in \mathbb{N}$. Recall that we have uniform Lipschitz estimates. So we find a convergent sequence of solutions to our approximating problems. The limit u is in $C^{0,1}(M^n) \cap C^\infty(M^n \setminus \partial M^n)$. $\square$

The rest of the article is organized as follows. We introduce supersolutions and some notation in Section 2. We mention C^0-estimates in Section 3. In Section 4, we prove uniform C^1-estimates. Then the C^2-estimates proved in Section 5 complete the a priori estimates and the proof of Theorem 1.2.

The author wants to thank Jürgen Jost and the Max Planck Institute for Mathematics in the Sciences for support and Guofang Wang for interesting discussions about the Schouten tensor.

2. Supersolutions and Notation

Before we define a supersolution, we explain more explicitly, how we regularize the equation. For fixed $k \in \mathbb{N}$ we take ψ_k such that

$$\psi_k(x) = \begin{cases} 0 & \operatorname{dist}(x, \partial M^n) < \frac{1}{k}, \\ 1 & \operatorname{dist}(x, \partial M^n) > \frac{2}{k} \end{cases}$$

and ψ_k is smooth with values in $[0, 1]$. Again, we drop the index k to keep the notation simple. We fix $\lambda \ge 0$ sufficiently large so that

$$\log\det(\underline{u}_{ij} + \psi \underline{u}_i \underline{u}_j - \tfrac{1}{2}\psi |\nabla \underline{u}|^2 g_{ij} + S_{ij} + \lambda(1-\psi) g_{ij}) \ge f(x, \underline{u}) \tag{2.1}$$

for any $\psi = \psi_k$, independent of k. As $\log\det(\cdot)$ is a concave function on positive definite matrices, (2.1) follows for k sufficiently large, if

$$\log\det(\underline{u}_{ij} + \underline{u}_i \underline{u}_j - \tfrac{1}{2} |\nabla \underline{u}|^2 g_{ij} + S_{ij}) \ge f(x, \underline{u}) \quad \text{on } M^n$$

and

$$\log\det(\underline{u}_{ij} + S_{ij} + \lambda g_{ij}) \ge f(x, \underline{u}) \quad \text{near } \partial M^n,$$

provided that the arguments of the determinants are positive definite.

We define

Definition 2.1 (supersolution)**.** A smooth function $\overline{u} : M^n \to \mathbb{R}$ is called a supersolution, if $\overline{u} \ge \underline{u}$ and for any ψ as considered above,

$$\log\det(\overline{u}_{ij} + \psi \overline{u}_i \overline{u}_j - \tfrac{1}{2}\psi |\nabla \overline{u}|^2 g_{ij} + S_{ij} + \lambda(1-\psi) g_{ij}) \le f(x, \underline{u})$$

holds for those points in M^n for which the tensor in the determinant is positive definite.

Lemma 2.2. *If M^n is a compact subdomain of flat $\mathbb{R}^n$, the subsolution $\underline{u}$ fulfills* (1.2) *and in addition*

$$\det(\underline{u}_{ij}) \ge s(x) e^{-2n\underline{u}} \det(g_{ij})$$

holds, where $\underline{u}_{ij} > 0$ in the matrix sense, then there exists a supersolution.

Proof. In flat $\mathbb{R}^n$, we have $S_{ij} = 0$. The inequality

$$\frac{\det(\underline{u}_{ij} + \psi \underline{u}_i \underline{u}_j - \frac{1}{2}\psi |\nabla \underline{u}|^2 g_{ij})}{e^{-2n\underline{u}} \det(g_{ij})} \geq s(x) \tag{2.2}$$

is fulfilled if ψ equals 0 or 1 by assumption. As above, (2.2) follows for any $\psi \in [0, 1]$. Thus (2.1) is fulfilled for $\lambda = 0$.

Let $\overline{u} = \sup\limits_{M^n} \underline{u} + 1 + \varepsilon |x|^2$ for $\varepsilon > 0$. It can be verified directly that $\overline{u}$ is a supersolution for $\varepsilon > 0$ fixed sufficiently small. $\square$

Our results can be extended to topologically more interesting manifolds, that may not allow for a globally defined convex function.

Remark 2.3. Assume that all assumptions of Theorem 1.2 are fulfilled, but the convex function χ is defined only in a neighborhood of the boundary. Then the conclusion of Theorem 1.2 remains true.

Proof. We have employed the globally defined convex function χ only to prove interior C^2-estimates for the regularized problems. On the set

$$\{x : \operatorname{dist}(x, \partial M^n) \geq \varepsilon\}, \quad \varepsilon > 0,$$

Theorem 1.3 implies C^2-estimates. In a neighborhood

$$U = \{x : \operatorname{dist}(x, \partial M^n) \leq 2\varepsilon\}$$

of the boundary, we can proceed as in the proof of Lemma 5.4. If the function W defined there attains its maximum over U at a point x in $\partial U \cap M^n$, i.e. $\operatorname{dist}(x, \partial M^n) = 2\varepsilon$, W is bounded and C^2-estimates follow, otherwise, we may proceed as in Lemma 5.4. $\square$

Notation. We set

$$\begin{aligned} w_{ij} =& u_{ij} + \psi u_i u_j - \tfrac{1}{2}\psi |\nabla u|^2 g_{ij} + S_{ij} + \lambda(1-\psi) g_{ij} \\ =& u_{ij} + \psi u_i u_j - \tfrac{1}{2}\psi |\nabla u|^2 g_{ij} + T_{ij} \end{aligned}$$

and use (w^{ij}) to denote the inverse of (w_{ij}). The Einstein summation convention is used. We lift and lower indices using the background metric. Vectors of length one are called directions. Indices, sometimes preceded by a semi-colon, denote covariant derivatives. We use indices preceded by a comma for partial derivatives. Christoffel symbols of the background metric are denoted by Γ^k_{ij}, so $u_{ij} = u_{;ij} = u_{,ij} - \Gamma^k_{ij} u_k$. Using the Riemannian curvature tensor (R_{ijkl}), we can interchange covariant differentiation

$$\begin{aligned} u_{ijk} =& u_{kij} + u_a g^{ab} R_{bijk}, \\ u_{iklj} =& u_{ikjl} + u_{ka} g^{ab} R_{bilj} + u_{ia} g^{ab} R_{bklj}. \end{aligned} \tag{2.3}$$

We write $f_z = \frac{\partial f}{\partial u}$ and $\operatorname{tr} w = w^{ij} g_{ij}$. The letter c denotes estimated positive constants and may change its value from line to line. It is used so that increasing c keeps the estimates valid. We use (c_j), (c^k), … to denote estimated tensors.

3. Uniform C^0-Estimates

The techniques of this section are quite standard, but they simplify the C^0-estimates used before for Schouten tensor equations, see [41, Proposition 3]. Here, we interpolate between the expressions for the Schouten tensors rather than between the functions inducing the conformal deformations.

We wish to show that we can apply the maximum principle or the Hopf boundary point lemma at a point, where a solution u touches the subsolution from above or the supersolution from below.

Note that u can touch $\overline{u}$ from below only in those points, where $\overline{u}$ is admissible. We did not assume that the upper barrier is admissible everywhere. But at those points, where it is not admissible, u cannot touch $\overline{u}$ from below. More precisely, at such a point, we have $\nabla u = \nabla\overline{u}$ and $D^2u \le D^2\overline{u}$. If $\overline{u}$ is not admissible there, we find $\xi\in\mathbb{R}^n$ such that $0 \ge (\overline{u}_{ij} + \psi\overline{u}_i\overline{u}_j - \frac12\psi|\nabla\overline{u}|g_{ij} + T_{ij})\xi^i\xi^j$. This implies that $0 \ge (u_{ij}+\psi u_iu_j-\frac12\psi|\nabla u|g_{ij}+T_{ij})\xi^i\xi^j$, so u is not admissible there, a contradiction. The idea, that the supersolution does not have to be admissible, appears already in [9].

Without loss of generality, we may assume that u touches $\underline{u}$ from above. Here, touching means $u = \underline{u}$ and $\nabla u = \nabla\underline{u}$ at a point, so our considerations include the case of touching at the boundary. It suffices to prove an inequality of the form

$$0 \le a^{ij}(\underline{u}-u)_{ij} + b^i(u-\underline{u})_i + d(\underline{u}-u) \tag{3.1}$$

with positive definite a^{ij}. The sign of d does not matter as we apply the maximum principle only at points, where u and $\underline{u}$ coincide.

Define

$$S^\psi_{ij}[v] = v_{ij} + \psi v_iv_j - \tfrac12\psi|\nabla v|^2g_{ij} + T_{ij}.$$

We apply the mean value theorem and get for a symmetric positive definite tensor a^{ij} and a function d

$$\begin{aligned}
0 \le{}& \log\det S^\psi_{ij}[\underline{u}] - \log\det S^\psi_{ij}[u] - f(x,\underline{u}) + f(x,u)\\
={}& \int_0^1 \frac{d}{dt}\log\det\Big\{tS^\psi_{ij}[\underline{u}] + (1-t)S^\psi_{ij}[u]\Big\}\,dt - \int_0^1 \frac{d}{dt} f(x, t\underline{u} + (1-t)u)dt\\
={}& a^{ij}((\underline{u}_{ij} + \psi\underline{u}_i\underline{u}_j - \tfrac12\psi|\nabla\underline{u}|^2g_{ij}) - (u_{ij} + \psi u_iu_j - \tfrac12\psi|\nabla u|^2g_{ij}))\\
&+ d\cdot(\underline{u}-u).
\end{aligned}$$

The first integral is well-defined as the set of positive definite tensors is convex. We have $|\nabla\underline{u}|^2 - |\nabla u|^2 = \langle\nabla(\underline{u}-u), \nabla(\underline{u}+u)\rangle$ and

$$\begin{aligned}
a^{ij}(\underline{u}_i\underline{u}_j - u_iu_j) ={}& a^{ij}\int_0^1 \frac{d}{dt}((t\underline{u}_i + (1-t)u_i)(t\underline{u}_j + (1-t)u_j))dt\\
={}& 2a^{ij}\int_0^1 (t\underline{u}_j + (1-t)u_j)dt\cdot(\underline{u}-u)_i,
\end{aligned}$$

so we obtain an inequality of the form (3.1). Thus, we may assume in the following that we have $\underline{u}\le u\le\overline{u}$.

4. Uniform C^1-Estimates

Lemma 4.1. *An admissible solution of* (1.4) *has uniformly bounded gradient.*

Proof. We apply a method similar to [38, Lemma 4.2]. Let

$$W = \tfrac{1}{2}\log|\nabla u|^2 + \mu u$$

for $\mu \gg 1$ to be fixed. Assume that W attains its maximum over M^n at an interior point x_0. This implies at x_0

$$0 = W_i = \frac{u^j u_{ji}}{|\nabla u|^2} + \mu u_i$$

for all i. Multiplying with u^i and using admissibility gives

$$\begin{aligned} 0 =& u^i u^j u_{ij} + \mu|\nabla u|^4 \\ \ge& -\psi|\nabla u|^4 + \tfrac{1}{2}\psi|\nabla u|^4 - c|\nabla u|^2 - \lambda|\nabla u|^2 + \mu|\nabla u|^4. \end{aligned}$$

The estimate follows for sufficiently large μ as λ, see (2.1), does not depend on ψ. If W attains its maximum at a boundary point x_0, we introduce normal coordinates such that W_n corresponds to a derivative in the direction of the inner unit normal. We obtain in this case $W_i = 0$ for $i < n$ and $W_n \le 0$ at x_0. As the boundary values of u and $\underline{u}$ coincide and $u \ge \underline{u}$, we may assume that $u_n \ge 0$. Otherwise, $0 \ge u_n \ge \underline{u}_n$ and $u_i = \underline{u}_i$, so a bound for $|\nabla u|$ follows immediately. Thus we obtain $0 \ge u^i W_i$ and the rest of the proof is identical to the case where W attains its maximum in the interior. □

Note that in order to obtain uniform C^1-estimates, we used admissibility, but did not differentiate (1.3).

5. C^2-Estimates

C^2-Estimates at the Boundary. Boundary estimates for an equation of the form $\det(u_{ij} + S_{ij}) = f(x)$ have been considered in [4]. It is straightforward to handle the additional term that is independent of u in the determinant and to use subsolutions like in [12, 13, 37, 38]. We want to point out that we were only able to obtain estimates for the second derivatives of u at the boundary by introducing ψ and thus removing gradient terms of u in the determinant near the boundary. The C^2-estimates at the boundary are very similar to [38]. We do not repeat the proofs for the double tangential and double normal estimates, but repeat that for the mixed tangential normal derivatives as we can slightly streamline this part. Our method does not imply uniform a priori estimates at the boundary as we look only at small neighborhoods of the boundary depending on the regularization or, more precisely, on the set, where $\psi = 0$.

Lemma 5.1 (Double Tangential Estimates)**.** *An admissible solution of* (1.4) *has uniformly bounded partial second tangential derivatives, i. e. for tangential directions τ_1 and τ_2, $u_{,ij}\tau_1^i\tau_2^j$ is uniformly bounded.*

Proof. This is identical to [38, Section 5.1], but can also be found at various other places. It follows directly by differentiating the boundary condition twice tangentially. □

All the remaining C^2-bounds depend on ψ.

Lemma 5.2 (Mixed Estimates). *For fixed ψ, an admissible solution of* (1.4) *has uniformly bounded partial second mixed tangential normal derivatives, i. e. for a tangential direction τ and for the inner unit normal ν, $u_{,ij}\tau^i\nu^j$ is uniformly bounded.*

Proof. The strategy of this proof is a follows. The differential operator T, defined below, differentiates tangentially along ∂M^n. We want to show that the normal derivative of Tu is bounded on ∂M^n. This implies a bound on mixed derivatives. To this end, we use an elliptic differential operator L that involves all higher order terms of the linearization of the equation. Thus, we can use the differentiated equation to bound LTu. Based on the subsolution $\underline{u}$, we construct a function $\vartheta \ge 0$ with $L\vartheta < 0$. Finally, we apply the maximum principle to

$$\Theta^\pm := A\vartheta + B|x - x_0|^2 \pm T(u - \underline{u})$$

with constants A, B. This implies that $\Theta^\pm \ge 0$ with equality at x_0. Thus, the normal derivative of Tu at x_0 is bounded.

This proof is similar to [38, Section 5.2]. The main differences are as follows. The modified definition of the linear operator T in (5.4) clarifies the relation between T and the boundary condition. The term T_{ij} does (in general) not vanish in a fixed boundary point for appropriately chosen coordinates. In [38], we could choose such coordinates. Similarly as in [38], we choose coordinates such that the Christoffel symbols become small near a fixed boundary point. Here, we can add and subtract the term T_{ij} in (5.7) as it is independent of u. Finally, we explain here more explicitly how to apply the inequality for geometric and arithmetic means in (5.9).

Fix normal coordinates around a point $x_0 \in \partial M^n$, so $g_{ij}(x_0)$ equals the Kronecker delta and the Christoffel symbols fulfill $\left|\Gamma^k_{ij}\right| \le c\,\mathrm{dist}(\cdot, x_0) = c|x - x_0|$, where the distance is measured in the flat metric using our chart, but is equivalent to the distance with respect to the background metric. Abbreviate the first $n-1$ coordinates by $\hat{x}$ and assume that M^n is locally given by $\{x^n \ge \omega(\hat{x})\}$ for a smooth function ω. We may assume that $(0, \omega(0))$ corresponds to the fixed boundary point x_0 and $\nabla\omega(0) = 0$. We restrict our attention to a neighborhood of x_0, $\Omega_\delta = \Omega_\delta(x_0) = M^n \cap B_\delta(x_0)$ for $\delta > 0$ to be fixed sufficiently small, where $\psi = 0$. Thus the equation takes the form

$$\log\det(u_{ij} + T_{ij}) = \log\det(u_{,ij} - \Gamma^k_{ij}u_k + T_{ij}) = f(x, u). \tag{5.1}$$

Assume furthermore that $\delta > 0$ is chosen so small that the distance function to ∂M^n is smooth in Ω_δ. The constant δ, introduced here, depends on ψ and tends to zero as the support of ψ tends to ∂M^n.

We differentiate the boundary condition tangentially

$$0 = (u - \underline{u})_{,t}(\hat{x}, \omega(\hat{x})) + (u - \underline{u})_{,n}(\hat{x}, \omega(\hat{x}))\omega_{,t}(\hat{x}), \quad t < n. \tag{5.2}$$

Differentiating (5.1) yields

$$w^{ij}(u_{,ijk} - \Gamma^l_{ij}u_{,lk}) = f_k + f_z u_k + w^{ij}(\Gamma^l_{ij,k}u_l - T_{ij,k}). \tag{5.3}$$

This motivates the definition of the differential operators T and L. Here $t < n$ is fixed and ω is evaluated at the projection of x to the first $n-1$ components

$$\begin{aligned} Tv &:= v_t + v_n\omega_t, \quad t < n, \\ Lv &:= w^{ij}v_{,ij} - w^{ij}\Gamma^l_{ij}v_l. \end{aligned} \tag{5.4}$$

On ∂M^n, we have $T(u - \underline{u}) = 0$, so we obtain

$$|T(u - \underline{u})| \le c(\delta) \cdot |x - x_0|^2 \quad \text{on } \partial\Omega_\delta. \tag{5.5}$$

As in [38, Section 5.2], [4, 12], we combine the definition of L, (5.4), and the differentiated Equation (5.3)

$$|LTu| \le c \cdot (1 + \operatorname{tr} w^{ij}) \quad in \Omega_\delta.$$

Derivatives of $\underline{u}$ are a priorily bounded, thus

$$|LT(u - \underline{u})| \le c \cdot (1 + \operatorname{tr} w^{ij}) \quad in \Omega_\delta. \tag{5.6}$$

Set $d := \operatorname{dist}(\cdot, \partial M^n)$, measured in the Euclidean metric of the fixed coordinates. We define for $1 \gg \alpha > 0$ and $\mu \gg 1$ to be chosen

$$\vartheta := (u - \underline{u}) + \alpha d - \mu d^2.$$

The function ϑ will be the main part of our barrier. As $\underline{u}$ is admissible, there exists $\varepsilon > 0$ such that

$$\underline{u}_{,ij} - \Gamma^l_{ij} \underline{u}_l + T_{ij} \ge 3\varepsilon g_{ij}.$$

We apply the definition of L

$$\begin{aligned} L\vartheta =& w^{ij}(u_{,ij} - \Gamma^l_{ij} u_l + T_{ij}) - w^{ij}(\underline{u}_{,ij} - \Gamma^l_{ij} \underline{u}_l + T_{ij}) \\ &+ \alpha w^{ij} d_{,ij} - \alpha w^{ij} \Gamma^l_{ij} d_l \\ &- 2\mu d w^{ij} d_{,ij} - 2\mu w^{ij} d_i d_j + 2\mu d w^{ij} \Gamma^l_{ij} d_l \end{aligned} \tag{5.7}$$

We have $w^{ij}(u_{,ij} - \Gamma^l_{ij} u_l + T_{ij}) = w^{ij} w_{ij} = n$. Due to the admissibility of $\underline{u}$, we get $-w^{ij}(\underline{u}_{,ij} - \Gamma^l_{ij} \underline{u}_l + T_{ij}) \le -3\varepsilon \operatorname{tr} w^{ij}$. We fix $\alpha > 0$ sufficiently small and obtain

$$\alpha w^{ij} d_{,ij} - \alpha w^{ij} \Gamma^l_{ij} d_l \le \varepsilon \operatorname{tr} w^{ij}.$$

Obviously, we have

$$-2\mu d w^{ij} d_{,ij} + 2\mu d w^{ij} \Gamma^l_{ij} d_l \le c\mu\delta \operatorname{tr} w^{ij}.$$

To exploit the term $-2\mu w^{ij} d_i d_j$, we use that $|d_i - \delta^n_i| \le c \cdot |x - x_0| \le c \cdot \delta$, so

$$-2\mu w^{ij} d_i d_j \le -\mu w^{nn} + c\mu\delta \max_{k,\, l} \left| w^{kl} \right|.$$

As w^{ij} is positive definite, we obtain by testing $\begin{pmatrix} w^{kk} & w^{kl} \\ w^{kl} & w_{ll} \end{pmatrix}$ with the vectors $(1, 1)$ and $(1, -1)$ that $|w^{kl}| \le \operatorname{tr} w^{ij}$. Thus (5.7) implies

$$L\vartheta \le -2\varepsilon \operatorname{tr} w^{ij} - \mu w^{nn} + c + c\mu\delta \operatorname{tr} w^{ij} \tag{5.8}$$

We may assume that $(w^{ij})_{i,\, j<n}$ is diagonal. Recall that our C^0-estimates imply that f is bounded. Thus

$$\begin{aligned} e^{-f} = \det(w^{ij}) =& \det \begin{pmatrix} w^{11} & 0 & \cdots & 0 & w^{1n} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & w^{n-1\, n-1} & w^{n-1\, n} \\ w^{1n} & \cdots & \cdots & w^{n-1\, n} & w^{nn} \end{pmatrix} \\ =& \prod_{i=1}^n w^{ii} \; - \; \sum_{i<n} \left| w^{ni} \right|^2 \; \prod_{\substack{j \ne i \\ j<n}} w^{jj} \; \le \; \prod_{i=1}^n w^{ii} \end{aligned} \tag{5.9}$$

implies that $\operatorname{tr} w^{ij}$ tends to infinity if w^{nn} tends to zero. So we can fix $\mu \gg 1$ such that the absolute constant in (5.8) can be absorbed. Note also that the geometric arithmetic means inequality implies

$$\tfrac{1}{n} \operatorname{tr} w^{ij} = \tfrac{1}{n} \sum_{i=1}^{n} w^{ii} \geq \Big(\prod_{i=1}^{n} w^{ii} \Big)^{1/n},$$

so (5.9) yields a positive lower bound for $\operatorname{tr} w^{ij}$. Finally, we fix $\delta = \delta(\mu)$ sufficiently small and use (5.8) to deduce that

$$L\vartheta \leq -\varepsilon \operatorname{tr} w^{ij}. \tag{5.10}$$

We may assume that δ is fixed so small that $\vartheta \geq 0$ in Ω_δ.

Define for A, $B \gg 1$ the function

$$\Theta^{\pm} := A\vartheta + B|x - x_0|^2 \pm T(u - \underline{u}).$$

Our estimates, especially (5.5) and (5.6), imply that $\Theta^{\pm} \geq 0$ on $\partial\Omega_\delta$ for $B \gg 1$, depending especially on $\delta(\psi)$, fixed sufficiently large and $L\Theta^{\pm} \leq 0$ in Ω_δ, when $A \gg 1$, depending also on B, is fixed sufficiently large. Thus the maximum principle implies that $\Theta^{\pm} \geq 0$ in Ω_δ. As $\Theta^{\pm}(x_0) = 0$, we deduce that $\Theta^{\pm}_{,n} \geq 0$, so we obtain a bound for $(Tu)_{,n}$ and the lemma follows. □

Lemma 5.3 (Double Normal Estimates)**.** *For fixed ψ, an admissible solution of* (1.4) *has uniformly bounded partial second normal derivatives, i. e. for the inner unit normal ν, $u_{,ij}\nu^i\nu^j$ is uniformly bounded.*

Proof. The proof is identical to [38, Section 5.3]. Note however, that the notation there is slightly different. There $-u_{,ij} + a_{ij}$ is positive definite instead of $u_{,ij} - \Gamma^k_{ij} u_k + T_{ij}$ here. □

Interior C^2-Estimates.

Lemma 5.4 (Interior Estimates)**.** *For fixed ψ, an admissible solution of* (1.4) *has uniformly bounded second derivatives.*

Proof. Note the admissibility implies that w_{ij} is positive definite. This implies a lower bound on the eigenvalues of u_{ij}.

For $\lambda \gg 1$ to be chosen sufficiently large, we maximize the functional

$$W = \log(w_{ij}\eta^i\eta^j) + \lambda\chi$$

over M^n and all (η^i) with $g_{ij}\eta^i\eta^j = 1$. Observe that W tends to infinity, if and only if $u_{ij}\eta^i\eta^j$ tends to infinity. We have

$$\begin{aligned} 2u_{ij}\eta^i\zeta^j =& 2w_{ij}\eta^i\zeta^j - 2(\psi u_i u_j - \tfrac{1}{2}\psi|\nabla u|^2 g_{ij} + T_{ij})\eta^i\zeta^j \\ \leq& w_{ij}\eta^i\eta^j + w_{ij}\zeta^i\zeta^j + c, \end{aligned}$$

so it suffices to bound terms of the form $w_{ij}\eta^i\eta^j$ from above. Thus, a bound on W implies a uniform C^2-bound on u.

In view of the boundary estimates obtained above, we may assume that W attains its maximum at an interior point x_0 of M^n. As in [8, Lemma 8.2] we may choose normal coordinates around x_0 and an appropriate extension of (η^i)

corresponding to the maximum value of W. In this way, we can pretend that w_{11} is a scalar function that equals $w_{ij}\eta^i\eta^j$ at x_0 and we obtain

$$0 = W_i = \frac{1}{w_{11}} w_{11;i} + \lambda\chi_i, \tag{5.11}$$

$$0 \geq W_{ij} = \frac{1}{w_{11}} w_{11;ij} - \frac{1}{w_{11}^2} w_{11;i} w_{11;j} + \lambda\chi_{ij} \tag{5.12}$$

in the matrix sense, $1 \leq i,\, j \leq n$. Here and below, all quantities are evaluated at x_0. We may assume that w_{ij} is diagonal and $w_{11} \geq 1$. Differentiating (1.4) yields

$$w^{ij} w_{ij;k} = f_k + f_z u_k, \tag{5.13}$$

$$w^{ij} w_{ij;11} - w^{ik} w^{jl} w_{ij;1} w_{kl;1} = f_{11} + 2f_{1z} u_1 + f_{zz} u_1 u_1 + f_z u_{11}. \tag{5.14}$$

Combining the convexity assumption on χ, (5.12) and (5.14) gives

$$\begin{aligned} 0 \geq& \frac{1}{w_{11}} w^{ij} w_{11;ij} - \frac{1}{w_{11}^2} w^{ij} w_{11;i} w_{11;j} + \lambda \operatorname{tr} w^{ij} \\ =& \frac{1}{w_{11}} w^{ij} (w_{11;ij} - w_{ij;11}) \\ &+ \frac{1}{w_{11}} w^{ik} w^{jl} w_{ij;1} w_{kl;1} - \frac{1}{w_{11}^2} w^{ij} w_{11;i} w_{11;j} \\ &+ \frac{1}{w_{11}} (f_{11} + 2f_{1z} u_1 + f_{zz} u_1 u_1 + f_z u_{11}) + \lambda \operatorname{tr} w^{ij}, \\ \equiv& \frac{1}{w_{11}} (P_4 + P_3 + R) + \lambda \operatorname{tr} w^{ij}, \end{aligned} \tag{5.15}$$

where

$$\begin{aligned} P_4 &= w^{ij} (w_{11;ij} - w_{ij;11}), \\ P_3 &= w^{ik} w^{jl} w_{ij;1} w_{kl;1} - \frac{1}{w_{11}} w^{ij} w_{11;i} w_{11;j}, \\ R &= f_{11} + 2f_{1z} u_1 + f_{zz} u_1 u_1 + f_z u_{11}. \end{aligned}$$

It will be convenient to decompose w_{ij} as follows

$$\begin{aligned} w_{ij} &= u_{ij} + r_{ij}, \\ r_{ij} &= \psi u_i u_j - \tfrac{1}{2}\psi |\nabla u|^2 g_{ij} + T_{ij}. \end{aligned} \tag{5.16}$$

The quantity r_{ij} is a priorily bounded, so the right-hand side of (5.14) is bounded from below by $-c(1 + w_{11})$,

$$R \geq -c \cdot (1 + w_{11}). \tag{5.17}$$

Let us first consider P_3. Recall that w_{ij} is diagonal and $w_{11} \geq w_{ii}$, $1 \leq i \leq n$. So we get $w^{jl} \geq \frac{1}{w_{11}} g^{jl}$. We also use (5.16) and the positive definiteness of w^{ij}

$$\begin{aligned}
P_3 =& w^{ik} w^{jl} w_{ij;1} w_{kl;1} - \frac{1}{w_{11}} w^{ij} w_{11;i} w_{11;j} \\
\geq& \frac{1}{w_{11}} w^{ij} (w_{i1;1} w_{j1;1} - w_{11;i} w_{11;j}) \\
=& \frac{1}{w_{11}} w^{ij} ((u_{i11} + r_{i1;1})(u_{j11} + r_{j1;1}) - (u_{11i} + r_{11;i})(u_{11j} + r_{11;j})) \\
\geq& \frac{1}{w_{11}} w^{ij} (u_{i11} u_{j11} - u_{11i} u_{11j} + 2u_{i11} r_{j1;1} - 2u_{11i} r_{11;j} - r_{11;i} r_{11;j}) \\
\equiv& P_{31} + P_{32} + P_{33},
\end{aligned} \tag{5.18}$$

where

$$\begin{gathered}
P_{31} = \frac{1}{w_{11}} w^{ij} (u_{i11} u_{j11} - u_{11i} u_{11j}), \\
P_{32} = \frac{2}{w_{11}} w^{ij} u_{i11} r_{j1;1}, \\
P_{33} = -\frac{2}{w_{11}} w^{ij} u_{11i} r_{11;j} - \frac{1}{w_{11}} w^{ij} r_{11;i} r_{11;j}.
\end{gathered}$$

We will bound P_{31}, P_{32}, and P_{33} individually. The term $r_{11;i}$ is of the form $c_i + c^k u_{ki}$ or, by (5.16), of the form $c_i + c^k w_{ki}$.

$$\begin{aligned}
P_{33} =& -2\frac{1}{w_{11}} w^{ij} u_{11i} r_{11;j} - \frac{1}{w_{11}} w^{ij} r_{11;i} r_{11;j} \\
=& -2\frac{1}{w_{11}} w^{ij} (w_{11i} - r_{11;i}) r_{11;j} - \frac{1}{w_{11}} w^{ij} r_{11;i} r_{11;j} \\
\geq& 2\lambda w^{ij} \chi_i r_{11;j} \quad \text{by (5.11)} \\
=& 2\lambda w^{ij} \chi_i (c_j + c^k w_{kj}) \\
\geq& -c\lambda (1 + \operatorname{tr} w^{ij}).
\end{aligned}$$

To estimate P_{32}, we use (2.3), (5.16), (5.11), $w^{ik} w_{kj} = \delta^i_j$, and the fact that $r_{j1;1}$ is of the form $c_j + \psi c_j w_{11} + c^k w_{kj}$

$$\begin{aligned}
P_{32} =& \frac{2}{w_{11}} w^{ij} (u_{11i} + u_a g^{ab} R_{b1i1}) r_{j1;1} \\
=& \frac{2}{w_{11}} w^{ij} (w_{11;i} - r_{11;i} + u_a g^{ab} R_{b1i1}) r_{j1;1} \\
=& -2\lambda w^{ij} \chi_i r_{j1;1} + \frac{2}{w_{11}} w^{ij} (-r_{11;i} + u_a g^{ab} R_{b1i1}) r_{j1;1} \\
=& -2\lambda w^{ij} \chi_i (c_j + \psi c_j w_{11} + c^k w_{kj}) \\
&+ \frac{2}{w_{11}} w^{ij} (c_i + c^k w_{ki})(c_j + \psi c_j w_{11} + c^k w_{kj}) \\
\geq& -c\lambda (1 + \operatorname{tr} w^{ij} + \psi w_{11} \operatorname{tr} w^{ij}) - c(1 + \operatorname{tr} w^{ij}).
\end{aligned}$$

It is crucial for the rest of the argument that the highest order error term contains a factor ψ. We interchange third covariant derivatives and get

$$P_{31} = \frac{1}{w_{11}} w^{ij} (u_{i11} u_{j11} - (u_{i11} + u_a g^{ab} R_{b11i})(u_{j11} + u_c g^{cd} R_{d11j}))$$

$$
\begin{aligned}
&\geq -2\frac{1}{w_{11}} w^{ij} u_{i11} u_a g^{ab} R_{b11j} - c\frac{1}{w_{11}} \operatorname{tr} w^{ij} \\
&= 2\lambda w^{ij} \chi_i u_a g^{ab} R_{b11j} + 2\frac{1}{w_{11}} w^{ij} r_{i1;1} u_a g^{ab} R_{b11j} - c\frac{1}{w_{11}} \operatorname{tr} w^{ij}
\end{aligned}
$$

by (5.11) and (5.16). Now, we obtain that

$$P_{31} \geq -c(1+\lambda)(1+\operatorname{tr} w^{ij}).$$

Recall that $\operatorname{tr} w^{ij}$ is bounded below by a positive constant. We employ (5.18) and get the estimate

$$\frac{1}{w_{11}} w^{ik} w^{jl} w_{ij;1} w_{kl;1} - \frac{1}{w_{11}^2} w^{ij} w_{11;i} w_{11;j} \geq -c(\lambda\psi + \frac{\lambda}{w_{11}}) \operatorname{tr} w^{ij}. \tag{5.19}$$

Next, we consider P_4. Equation (2.3) implies

$$
\begin{aligned}
u_{11ij} =& u_{ij11} + u_{a1} g^{ab} R_{bi1j} + u_a g^{ab} R_{bi1j;1} + u_{1a} g^{ab} R_{bij1} + u_{ia} g^{ab} R_{b1j1} \\
&+ u_{aj} g^{ab} R_{b11i} + u_a g^{ab} R_{b11i;j} \\
\geq& u_{ij11} - c_{ij}(1+w_{11}).
\end{aligned}
$$

We use (5.16)

$$
\begin{aligned}
w^{ij}(w_{11;ij} - w_{ij;11}) =& w^{ij}(u_{11ij} - u_{ij11}) + w^{ij}(r_{11;ij} - r_{ij;11}) \\
\geq& w^{ij}(r_{11;ij} - r_{ij;11}) - c w_{11} \operatorname{tr} w^{ij} \\
=& w^{ij}(\psi_{ij} u_1^2 + 4\psi_i u_1 u_{1j} + 2\psi u_{1j} u_{1i} + 2\psi u_1 u_{1ij}) \\
&+ w^{ij}(-\psi_{11} u_i u_j - 4\psi_1 u_{i1} u_j - 2\psi u_{1i} u_{1j} - 2\psi u_i u_{j11}) \\
&+ w^{ij}(-\tfrac{1}{2}\psi_{ij} |\nabla u|^2 g_{11} - 2\psi_i u^k u_{kj} g_{11} - \psi u_j^k u_{ki} g_{11} - \psi u^k u_{kij} g_{11}) \\
&+ w^{ij}(\tfrac{1}{2}\psi_{11} |\nabla u|^2 g_{ij} + 2\psi_1 u^k u_{k1} g_{ij} + \psi u_1^k u_{k1} g_{ij} + \psi u^k u_{k11} g_{ij}) \\
&+ w^{ij}(T_{11;ij} - T_{ij;11}) - c w_{11} \operatorname{tr} w^{ij} \\
=& P_{41} + P_{42} - c w_{11} \operatorname{tr} w^{ij},
\end{aligned}
$$

where

$$
\begin{aligned}
P_{41} =& w^{ij}(\psi_{ij} u_1^2 + 4\psi_i u_1 u_{1j} + 2\psi u_{1j} u_{1i}) \\
&+ w^{ij}(-\psi_{11} u_i u_j - 4\psi_1 u_{i1} u_j - 2\psi u_{1i} u_{1j}) \\
&+ w^{ij}(-\tfrac{1}{2}\psi_{ij} |\nabla u|^2 g_{11} - 2\psi_i u^k u_{kj} g_{11}) \\
&+ w^{ij}(\tfrac{1}{2}\psi_{11} |\nabla u|^2 g_{ij} + 2\psi_1 u^k u_{k1} g_{ij}) \\
&+ w^{ij}(T_{11;ij} - T_{ij;11}),
\end{aligned}
$$

and

$$
\begin{aligned}
P_{42} =& w^{ij}(2\psi u_1 u_{1ij} - 2\psi u_i u_{j11} - \psi u^k u_{kij} g_{11} + \psi u^k u_{k11} g_{ij}) \\
&+ w^{ij}(-\psi u_j^k u_{ki} g_{11} + \psi u_1^k u_{k1} g_{ij}).
\end{aligned}
$$

The last term in the first line and the last term in the second line of the definition of P_{41} cancel. Note once more, that

$$w^{ij} u_{jk} = w^{ij}(w_{jk} - r_{jk}) = \delta_k^i - w^{ij} r_{jk}.$$

Moreover, w_{ij} is positive definite, diagonal, and $w_{11} \geq w_{ii}$, $1 \leq i \leq n$, so $|w_{ij}| \leq w_{11}$ for any $1 \leq i, j \leq n$. We obtain

$$P_{41} \geq -cw_{11} \operatorname{tr} w^{ij}.$$

Note that this constant depends on derivatives of ψ. So our estimate does also depend on ψ. We interchange covariant third derivatives (2.3) and employ once again (5.16)

$$\begin{aligned}
w^{ij}(w_{11;ij} - w_{ij;11}) \geq & w^{ij}(2\psi u_1 u_{1ij} - 2\psi u_i u_{j11} - \psi u^k u_{kij} g_{11} + \psi u^k u_{k11} g_{ij}) \\
& + w^{ij}(-\psi u_j^k u_{ki} g_{11} + \psi u_1^k u_{k1} g_{ij}) - cw_{11} \operatorname{tr} w^{ij} \\
= & 2\psi u_1 w^{ij} u_{ij1} + 2\psi u_1 w^{ij} u_a g^{ab} R_{bi1j} \\
& - \psi g_{11} u^k w^{ij} u_{ijk} - \psi g_{11} u^k w^{ij} u_a g^{ab} R_{bikj} \\
& - 2\psi u_i w^{ij} u_{11j} - 2\psi u_i w^{ij} u_a g^{ab} R_{b1j1} \\
& + \psi u^k u_{11k} \operatorname{tr} w^{ij} + \psi u^k u_a g^{ab} R_{b1k1} \operatorname{tr} w^{ij} \\
& - \psi g_{11} w^{ij} (w_{ik} - r_{ik})(w_{jl} - r_{jl}) g^{kl} \\
& + \psi (w_{1k} - r_{1k})(w_{1l} - r_{1l}) g^{kl} \operatorname{tr} w^{ij} - cw_{11} \operatorname{tr} w^{ij} \\
\geq & P_{43} + P_{44} - cw_{11} \operatorname{tr} w^{ij},
\end{aligned}$$

where

$$\begin{aligned}
P_{43} &= 2\psi u_1 w^{ij} u_{ij1} - \psi g_{11} u^k w^{ij} u_{ijk} - 2\psi u_i w^{ij} u_{11j} + \psi u^k u_{11k} \operatorname{tr} w^{ij}, \\
P_{44} &= -\psi g_{11} w^{ij} (w_{ik} - r_{ik})(w_{jl} - r_{jl}) g^{kl} + \psi (w_{1k} - r_{1k})(w_{1l} - r_{1l}) g^{kl} \operatorname{tr} w^{ij}.
\end{aligned}$$

As above, we see that

$$P_{44} \geq \psi w_{11}^2 \operatorname{tr} w^{ij} - cw_{11} \operatorname{tr} w^{ij}.$$

We continue to estimate P_4 and replace third derivatives of u by derivatives of w_{ij}. Equations (5.13) and (5.11) allow us to replace these terms by terms involving at most second derivatives of u

$$\begin{aligned}
& w^{ij}(w_{11;ij} - w_{ij;11}) \\
& \geq 2\psi u_1 w^{ij} w_{ij;1} - 2\psi u_1 w^{ij} r_{ij;1} - \psi g_{11} u^k w^{ij} w_{ij;k} + \psi g_{11} u^k w^{ij} r_{ij;k} \\
& \quad - 2\psi u_i w^{ij} w_{11;j} + 2\psi u_i w^{ij} r_{11;j} + \psi u^k w_{11;k} \operatorname{tr} w^{ij} - \psi u^k r_{11;k} \operatorname{tr} w^{ij} \\
& \quad + \psi w_{11}^2 \operatorname{tr} w^{ij} - cw_{11} \operatorname{tr} w^{ij} \\
& \geq -2\psi u_i w^{ij} w_{11;j} + \psi u^k w_{11;k} \operatorname{tr} w^{ij} + \psi w_{11}^2 \operatorname{tr} w^{ij} - cw_{11} \operatorname{tr} w^{ij} \\
& \geq 2\lambda\psi w_{11} w^{ij} u_i \chi_j - \lambda\psi w_{11} u^k \chi_k \operatorname{tr} w^{ij} + \psi w_{11}^2 \operatorname{tr} w^{ij} - cw_{11} \operatorname{tr} w^{ij} \\
& \geq -c\lambda\psi w_{11} \operatorname{tr} w^{ij} + \psi w_{11}^2 \operatorname{tr} w^{ij} - cw_{11} \operatorname{tr} w^{ij}.
\end{aligned}$$

This gives

$$\frac{1}{w_{11}} w^{ij}(w_{11;ij} - w_{ij;11}) \geq -c\lambda\psi \operatorname{tr} w^{ij} + \psi w_{11} \operatorname{tr} w^{ij} - c \operatorname{tr} w^{ij}. \tag{5.20}$$

We estimate the respective terms in (5.15) using (5.17), (5.19), and (5.20) and obtain

$$0 \geq \left\{ \psi(w_{11} - c\lambda) + (\lambda - c - \frac{c\lambda}{w_{11}}) \right\} \operatorname{tr} w^{ij}. \tag{5.21}$$

Recall once more, that $c = c(\psi, \dots)$ depends on the regularization.

Assume that all c's in (5.21) are equal. Now we fix λ equal to $c+1$. Then (5.21) implies that w_{11} is bounded above. □

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