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EXISTENCE OF SOLUTIONS FOR A DEGENERATE SEAWATER INTRUSION PROBLEM

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Abstract. We study a seawater intrusion problem in a confined aquifer. This process can be formulated as a coupled system of partial differential equations which includes an elliptic and a degenerate parabolic equation. Existence results of weak solutions, under realistic assumptions, are established through time discretization combined with parabolic regularization.

1. Introduction

The motivation for the following mathematical problem arises from the area of modelling groundwater in coastal aquifers. Groundwater is a major source of water supply in many parts of the world. It supports domestic consumption, irrigation, and industrial processing. The use of groundwater has been rising steadily in the last several decades. It has been exploited to sustain a growing population and economy. The loss of surface water to pollution has further increased the stress on groundwater extraction. By now, as much as on third of the world's drinking water is derived from groundwater. Although better protected than surface water, groundwater can also be contaminated. Once contaminated–because of its subsurface, hidden, and inaccessible nature–detection and remediation are more difficult.

Despite its abundance, unregulated extraction of groundwater can easily cause localized problems. In coastal zones, the intensive extraction of groundwater has upset the long established balance between freshwater and seawater potentials, causing encroachment of seawater into freshwater aquifers. As a large proportion of the world's population (about 70%) dwells in coastal zones, the optimal exploitation of fresh groundwater and the control of seawater intrusion are the challenges for the present-day and future water supply engineers and managers. The modelling of groundwater in coastal aquifers is an important and difficult issue in water resources. The primary difficulty resides in efficient and accurate simulation of the movement of the saltwater front. Freshwater and saltwater are miscible fluids and therefore, the zone separating them takes the form of a transition zone caused by hydrodynamic dispersion. For certain problems, the simulation can be simplified by assuming that each liquid is confined to a well defined portion of the flow domain with an abrupt interface separating the two domains (cf. [5], [6] and [11]). This

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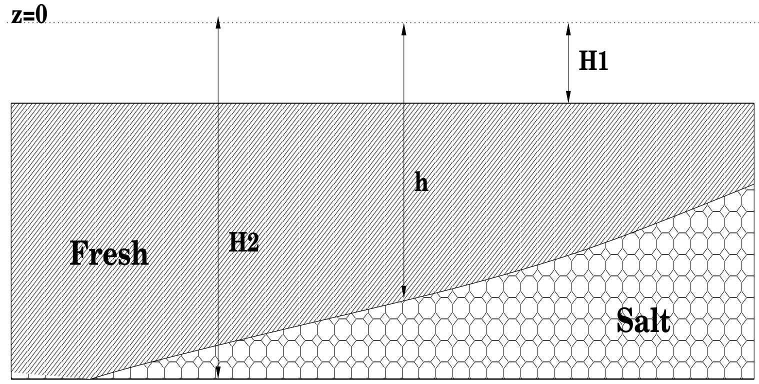

FIGURE 2.1. Saltwater intrusion phenomena

modelling approach, called sharp interface, does not give information concerning the nature of the transition zone but does reproduce the regional flow dynamics of the system and the response of the interface to applied stresses, for more details about seawater intrusion problem with sharp interface approach we refer to [6, Section 13.2].

In the present paper, we address the seawater intrusion problem with sharp interface model in a confined aquifer. The model to be presented herein is formulated in terms of a two-dimensional coupled system consisting of an elliptic and a degenerate parabolic equations. The main difficulties related to the analysis of this system are the coupling between equations and the degeneracy due to the possibility to have no saltwater in some zones of the aquifer. This type of system occurs in a variety of physical situations such as in petroleum engineering and has been studied by many authors (see, e.g., [1, 2, 3, 4, 13, 9, 12]). Let us also mention that the steady state seawater problem has been treated in [10], however the inflow of the saltwater is mostly a transient process, then the time-dependent problem is of greater practical interest. In the present paper, we use the technique developed by Alt and Luckauss [1] to derive an existence result for the transient system modeling seawater interface problem with sharp interface under realistic assumptions.

The outline of the paper is as follows. In section 2 all necessary mathematical notations are defined, the equations of the problem are formulated and the general assumptions are stated. The third section is devoted to the presentation and analysis of a regularized problem. We prove the existence of at least one weak solution for the problem in the non degenerate case. The result is obtained by time discretization and the technique developed in [1]. In the last section, we get the existence of weak solutions for the degenerate case.

2. Problem setting and assumptions

The differential system. We consider the flow of fresh and salt groundwater, separated by a sharp interface, in a confined aquifer. The aquifer is bounded by two approximately horizontal and impermeable layers (Figure 2.1). The lower and upper surfaces of the aquifer are described by $z = -H_2$ and $z = -H_1$, respectively.

The substitution of Darcy's law into continuity equations of the two fluids (fresh and salt), the continuity of flux and the pressure through the interfacial boundary and the integral of aquations over the vertical lead to the following system of coupled partial differential equations [6]:

$$\begin{gathered} S(x)\partial_t h - \operatorname{div}(\alpha k(x)T_s(h)\nabla h) + \operatorname{div}(k(x)T_s(h)\nabla\varphi) = -I_s \\ -\operatorname{div}(k(x)T_a\nabla\varphi) + \operatorname{div}(\alpha k(x)T_s(h)\nabla h) = I_f + I_s \end{gathered} \tag{2.1}$$

for $(x,t) \in \Omega_T := \Omega \times J$ with $J =]0,T[$ and Ω is an open bounded domain of $\mathbb{R}^2$, describing the projection of the porous medium on the horizontal plane $z = 0$, with a smooth boundary $\Gamma = \Gamma_D \cup \Gamma_N$. Here $T_a = H_2 - H_1$ is the thickness of the aquifer, $T_s = H_2 - h$ is the thickness of saltwater zone, k is the hydraulic conductivity, S is the storativity of the aquifer, α is a positive constant representing the relative density difference, φ is the freshwater hydraulic head, h is the depth of the interface and where I_f and I_s are supply functions, representing distributed surface supply of fresh and saline water into the aquifer.

Introducing the new variables $f = \dfrac{\varphi}{\alpha}$ and $K = \alpha k$ leads to the following system:

$$\begin{gathered} S(x)\partial_t h - \operatorname{div}(K(x)T_s(h)\nabla h) + \operatorname{div}(K(x)T_s(h)\nabla f) = -I_s \\ -\operatorname{div}(K(x)T_a\nabla f) + \operatorname{div}(K(x)T_s(h)\nabla h) = I_f + I_s \end{gathered} \tag{2.2}$$

The boundary conditions are

$$\begin{gathered} h = h_D,\ f = f_D \quad \text{on } \Gamma_D \\ (K(x)T_s(h)\nabla h - K(x)T_s(h)\nabla f)\cdot\vec{n} = 0 \quad \text{on } \Gamma_N \\ (K(x)T_a\nabla f - K(x)T_s(h)\nabla h)\cdot\vec{n} = 0 \quad \text{on } \Gamma_N \end{gathered} \tag{2.3}$$

where f_D and h_D are given functions, and $\vec{n}$ is the outward unit normal to Γ. The initial condition is

$$h(x,0) = h_0(x), \quad x \in \Omega. \tag{2.4}$$

Notation and assumptions. We introduce the Hilbert space

$$V = \{\varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma_D\},$$

under assumption (A1) below, the norm and semi-norm defined on $H^1(\Omega)$ are equivalent in V. We denote by V' the dual space of V and by $\langle .,. \rangle$ the duality pairing between V and V'. $(.,.)_Q$ is the $L^2(Q)$ inner product (Q is omitted if $Q = \Omega$). δ is a small positive real number and we denote by $T_f(h) = h - H_1 = T_a - T_s(h)$ the thickness of freshwater zone. We make now the following assumptions:

(A1) $\Omega \subset \mathbb{R}^2$ is an open bounded domain with Lipschitz boundary Γ,

$$\Gamma = \Gamma_D \cup \Gamma_N, \quad \Gamma_D \cap \Gamma_N = \emptyset, \quad \text{and} \quad \operatorname{meas}(\Gamma_D) \neq 0.$$

(A2) $S = S(x) \in L^\infty(\Omega)$, $S(x) \geq S_* > 0$, and $K(x)$ is bounded, symmetric, and uniformly positive definite matrix, i.e.,

$$0 < K_* \leq |\xi|^{-2} \sum_{i,j=1}^{2} K_{i,j}(x)\xi_i\xi_j \leq K^* < \infty \quad x \in \Omega,\ \xi \neq 0 \in \mathbb{R}^2.$$

(A3) H_1, H_2 are positive constants such that $H_2 > H_1 + \delta$.

(A4) $I_s \geq 0$, $(T_a - \delta)I_f - \delta I_s \geq 0$, $I_s \in L^\infty(J; L^2(\Omega))$, and $I_f \in L^\infty(J; V'(\Omega))$.

(A5) The boundary data satisfy f_D, $h_D \in L^2(J; H^1(\Omega))$,

$$\partial_t h_D \in L^1(\Omega_T), \quad S(x)\partial_t h_D \in L^2(J, V'), \quad H_1 + \delta \leq h_D \leq H_2$$

a.e. on Ω_T.

(A6) h_0 satisfies $H_1 + \delta \le h_0 \le H_2$, a.e. on Ω.

To study the system (2.2)-(2.4), we use the regularization technique described above.

3. The regularized problem

To solve problem (2.2)-(2.4), we first consider its parabolic regularization:

$$\begin{gathered} S(x)\partial_t h_\varepsilon - \operatorname{div}(K(x)T_s(h_\varepsilon)\nabla h_\varepsilon) - \varepsilon\Delta h_\varepsilon + \operatorname{div}(K(x)T_s(h_\varepsilon)\nabla f_\varepsilon) = -I_s \\ -\operatorname{div}(K(x)T_a\nabla f_\varepsilon) + \operatorname{div}(K(x)T_s(h_\varepsilon)\nabla h_\varepsilon) = I_f + I_s, \end{gathered} \tag{3.1}$$

$$\begin{gathered} h_\varepsilon = h_D, \quad f_\varepsilon = f_D \quad \text{on } \Gamma_D \\ (K(x)T_s(h_\varepsilon)\nabla h_\varepsilon + \varepsilon\nabla h_\varepsilon - K(x)T_s(h_\varepsilon)\nabla f_\varepsilon)\cdot\overrightarrow{n} = 0 \quad \text{on } \Gamma_N \\ (K(x)T_a\nabla f_\varepsilon - K(x)T_s(h_\varepsilon)\nabla h_\varepsilon)\cdot\overrightarrow{n} = 0 \quad \text{on } \Gamma_N \end{gathered} \tag{3.2}$$

and

$$h_\varepsilon(x,0) = h_0(x), \quad x \in \Omega. \tag{3.3}$$

where ε is a small positive parameter.

Definition 3.1. A pair of functions $(h_\varepsilon, f_\varepsilon)$, is called a weak solution to the regularized problem $(3.1)-(3.3)$ if it satisfies the system:

$$H_1 + \delta \le h_\varepsilon \le H_2, \quad \text{a.e. on } \Omega_T; \tag{3.4}$$

$$h_\varepsilon \in L^2(J,V) + h_D, \quad S(x)\partial_t h_\varepsilon \in L^2(J,V'), \quad f_\varepsilon \in L^2(J,V) + f_D, \tag{3.5}$$

$$-\int_J \langle S\partial_t h_\varepsilon, \varphi\rangle dt = \int_J (Sh_\varepsilon, \frac{\partial\varphi}{\partial t}) + (S(x)h_0(x), \varphi(x,0)) \quad \forall \varphi \in \mathcal{D}(\Omega\times[0,T[), \tag{3.6}$$

$$\begin{aligned} &\int_J \langle S\partial_t h_\varepsilon, v\rangle dt + \int_J (K(x)T_s(h_\varepsilon)\nabla h_\varepsilon, \nabla v)dt \\ &+\varepsilon\int_J (\nabla h_\varepsilon, \nabla v) - \int_J (K(x)T_s(h_\varepsilon)\nabla f_\varepsilon, \nabla v)dt = -\int_J (I_s, v)dt, \quad \forall v \in L^2(0,T,V), \end{aligned} \tag{3.7}$$

$$\begin{aligned} \int_J (K(x)T_a\nabla f_\varepsilon, \nabla w)dt &- \int_J (K(x)T_s(h_\varepsilon)\nabla h_\varepsilon, \nabla w)dt \\ &= \int_J (I_s + I_f, w)dt, \quad \forall w \in L^2(0,T,V)\,. \end{aligned} \tag{3.8}$$

We now state the main result of this section.

Theorem 3.2. *Under assumptions (A1)–(A6), the system* (3.4)-(3.8) *has a weak solution in the sense of definition 3.1.*

To show this proposition we make use of a backward time difference scheme:

For each positive integer M, divide J into $m = 2^M$ subintervals of equal length $\Delta t = T/m = 2^{-M}T$. Set $t_i = i\Delta t$ and $J_i = (t_{i-1}, t_i]$ for an integer i, $1 \le i \le m$. Denote the time difference operator by

$$\partial^\eta v(t) = \frac{v(t+\eta) - v(t)}{\eta},$$

for any function $v(t)$ and constant $\eta \in \mathbb{R}$. Also we define

$$l_{\Delta t}(V) = \{v \in L^\infty(J;V) : v \text{ is constant in time on each subinterval } J_i \subset J\}.$$

For $v_{\Delta t} \in l_{\Delta t}(V)$, set $v^i = v_{\Delta t}|_{J_i}$ for notational convenience. Finally, let

$$h_{D,\Delta t} = \frac{1}{\Delta t}\int_{J_i} h_D(x,\tau)d\tau, \quad f_{D,\Delta t} = \frac{1}{\Delta t}\int_{J_i} f_D(x,\tau)d\tau, \quad t \in J_i.$$

Now the discrete time solution is a pair of functions $h_{\Delta t} \in l_{\Delta t}(V) + h_{D,\Delta t}$, $f_{\Delta t} \in l_{\Delta t}(V) + f_{D,\Delta t}$ satisfying

$$H_1 + \delta \le h_{\Delta t} \le H_2, \quad \text{a.e. on } \Omega_T. \tag{3.9}$$

$$\begin{aligned} &\int_J (S\partial^{-\Delta t} h_{\Delta t}, v)dt + \int_J (K(x)T_s(h_{\Delta t})\nabla h_{\Delta t}, \nabla v)dt \\ &+ \varepsilon\int_J (\nabla h_{\Delta t}, \nabla v)dt - \int_J (K(x)T_s(h_{\Delta t})\nabla f_{\Delta t}, \nabla v)dt \\ &= -\int_J (I_s, v)dt \quad \forall v \in l_{\Delta t}(V), \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\int_J (K(x)T_a\nabla f_{\Delta t}, \nabla w)dt - \int_J (K(x)T_s(h_{\Delta t})\nabla h_{\Delta t}, \nabla w)dt \\ &= \int_J (I_s + I_f, w)dt, \quad \forall w \in l_{\Delta t}(V). \end{aligned} \tag{3.11}$$

This approximation scheme is extended such that $h_{\Delta t} = h_0$ for $t \le 0$.

In the following, C indicates a generic constant independent of Δt which will probably take different values in different occurrences.

Lemma 3.3. *The discrete scheme has at least one solution* $(h_{\Delta t}, f_{\Delta t})$.

The proof of this lemma will be given in the end of this section.

Lemma 3.4. *The solution of the discrete schemes also satisfies*

$$\varepsilon\int_J \|h_{\Delta t}\|^2_{H^1(\Omega)} + \int_J \|f_{\Delta t}\|^2_{H^1(\Omega)} \le C \tag{3.12}$$

with a constant C independent of Δt.

Proof. Taking $v = h_{\Delta t} - h_{D,\Delta t} \in l_{\Delta t}(V)$ in (3.10) and $w = f_{\Delta t} - f_{D,\Delta t} \in l_{\Delta t}(V)$ in (3.11), we have

$$\begin{aligned} &\int_J (S\partial^{-\Delta t} h_{\Delta t}, h_{\Delta t} - h_{D,\Delta t}) + \int_J (K(x)T_s(h_{\Delta t})\nabla h_{\Delta t}, \nabla h_{\Delta t} - \nabla h_{D,\Delta t}) \\ &+ \varepsilon\int_J (\nabla h_{\Delta t}, \nabla h_{\Delta t} - \nabla h_{D,\Delta t}) - \int_J (K(x)T_s(h_{\Delta t})\nabla f_{\Delta t}, \nabla h_{\Delta t} - \nabla h_{D,\Delta t}) \\ &= -\int_J (I_s, h_{\Delta t} - h_{D,\Delta t}), \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &\int_J (K(x)T_a\nabla f_{\Delta t}, \nabla f_{\Delta t} - \nabla f_{D,\Delta t})dt - \int_J (K(x)T_s(h_{\Delta t})\nabla h_{\Delta t}, \nabla f_{\Delta t} - \nabla f_{D,\Delta t})dt \\ &= \int_J (I_s + I_f, f_{\Delta t} - f_{D,\Delta t})dt, \end{aligned} \tag{3.14}$$

Summing these two equalities and noting that $T_a = T_s(h_{\Delta t}) + T_f(h_{\Delta t}))$, we have

$$\begin{aligned}
&\int_J (S\partial^{-\Delta t} h_{\Delta t}, h_{\Delta t})dt + \varepsilon \int_J (\nabla h_{\Delta t}, \nabla h_{\Delta t})\\
&+ \int_J (K(x)T_f(h_{\Delta t})\nabla f_{\Delta t}, \nabla f_{\Delta t})dt + \int_J (K(x)T_s(h_{\Delta t})\nabla h_{\Delta t}, \nabla h_{\Delta t})dt\\
&- 2\int_J (K(x)T_s(h_{\Delta t})\nabla f_{\Delta t}, \nabla h_{\Delta t})dt + \int_J (K(x)T_s(h_{\Delta t})\nabla f_{\Delta t}, \nabla f_{\Delta t})dt\\
&= \int_J (S\partial^{-\Delta t} h_{\Delta t}, h_{D,\Delta t})dt + \varepsilon \int_J (\nabla h_{\Delta t}, \nabla h_{D,\Delta t})\\
&\quad + \int_J (K(x)T_f(h_{\Delta t})\nabla f_{\Delta t}, \nabla f_{D,\Delta t})dt\\
&\quad + \int_J (K(x)T_s(h_{\Delta t})\nabla h_{\Delta t}, \nabla h_{D,\Delta t})dt + \int_J (K(x)T_s(h_{\Delta t})\nabla f_{\Delta t}, \nabla f_{D,\Delta t})dt\\
&\quad - \int_J (K(x)T_s(h_{\Delta t})\nabla f_{\Delta t}, \nabla h_{D,\Delta t})dt - \int_J (K(x)T_s(h_{\Delta t})\nabla h_{\Delta t}, \nabla f_{D,\Delta t})dt\\
&\quad - \int_J (I_s, h_{\Delta t} - h_{D,\Delta t}) + \int_J (I_s + I_f, f_{\Delta t} - f_{D,\Delta t})dt
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_J (S\partial^{-\Delta t} h_{\Delta t}, h_{\Delta t})dt + \varepsilon \int_J (\nabla h_{\Delta t}, \nabla h_{\Delta t}) + \int_J (K(x)T_f(h_{\Delta t})\nabla f_{\Delta t}, \nabla f_{\Delta t})dt\\
&+ \int_J (K(x)T_s(h_{\Delta t})\nabla(h_{\Delta t} - f_{\Delta t}), \nabla(h_{\Delta t} - f_{\Delta t}))dt\\
&= \int_J (S\partial^{-\Delta t} h_{\Delta t}, h_{D,\Delta t})dt + \varepsilon \int_J (\nabla h_{\Delta t}, \nabla h_{D,\Delta t})\\
&\quad + \int_J (K(x)T_f(h_{\Delta t})\nabla f_{\Delta t}, \nabla f_{D,\Delta t})dt\\
&+ \int_J (K(x)T_s(h_{\Delta t})\nabla(h_{\Delta t} - f_{\Delta t}), \nabla(h_{D,\Delta t} - f_{D,\Delta t}))dt\\
&- \int_J (I_s, h_{\Delta t} - h_{D,\Delta t})dt + \int_J (I_s + I_f, f_{\Delta t} - f_{D,\Delta t})dt
\end{aligned}$$

Next it is easy to see that

$$\int_J (S\partial^{-\Delta t} h_{\Delta t}, h_{\Delta t})dt = \sum_{i=1}^{i=m} (S(h^i - h^{i-1}), h^i) \geq \frac{1}{2}\{(Sh^m, h^m) - (Sh^0, h^0)\}. \quad (3.15)$$

Also, since $S = S(x) \in L^\infty(\Omega)$, $H_1 + \delta \leq h_{\Delta t} \leq H_2$ and $H_1 + \delta \leq h_D \leq H_2$, we have

$$\begin{aligned}
\int_J (S\partial^{-\Delta t} h_{\Delta t}, h_{D,\Delta t})dt &= (Sh^m, h_D^m) - (Sh^0, h_D^0) - \int_0^{T-\Delta t} (Sh_{\Delta t}, \partial^{\Delta t} h_{D,\Delta t})dt,\\
&\leq C + C\int_0^{T-\Delta t} \|\partial^{\Delta t} h_{D,\Delta t}\|_{L^1(\Omega)},
\end{aligned} \quad (3.16)$$

and from

$$\int_0^{T-\Delta t} \|\partial^{\Delta t} h_{D,\Delta t}\|_{L^1(\Omega)} = \sum_{i=1}^{m-1} \|h_D^i - h_D^{i-1}\|_{L^1(\Omega)},$$

it follows that

$$\int_0^{T-\Delta t} \|\partial^{\Delta t} h_{D,\Delta t}\|_{L^1(\Omega)} = \sum_{i=1}^{m-1} \frac{1}{\Delta t} \| \int_{t_{i-1}}^{t_i} \int_{t-\Delta t}^{t} \partial_t h_D(.,\tau) d\tau dt \|_{L^1(\Omega)},$$

so

$$\int_0^{T-\Delta t} \|\partial^{\Delta t} h_{D,\Delta t}\|_{L^1(\Omega)} \le \int_J \|\partial_t h_D\|_{L^1(\Omega)} dt. \tag{3.17}$$

Now, we use Young inequality and combine (A2), (A5), (3.15)-(3.17) to taht for every $\mu > 0$,

$$\begin{aligned}\varepsilon \int_J (\nabla h_{\Delta t}, \nabla h_{\Delta t}) + \int_J ((K(x) T_f(h_{\Delta t}) - 2\mu C)\nabla f_{\Delta t}, \nabla f_{\Delta t}) dt \\ + \int_J (K(x) T_s(h_{\Delta t}) \nabla (h_{\Delta t} - f_{\Delta t}), \nabla (h_{\Delta t} - f_{\Delta t})) dt \le C\end{aligned}$$

Then for μ small enough and by the fact that $T_f(h_{\Delta t}) \ge \delta > 0$ (for all $h_{\Delta t}$ satisfying (3.9)) and K satisfying (A1) we obtain the desired result. □

Lemma 3.5. *There is a subsequence such that, $h_{\Delta t}$ strongly converges in $L^2(\Omega_T)$.*

Proof. According to [9, Lemma 2.6], it suffices to show that there exists a constant C such that, for any $\xi > 0$,

$$\frac{1}{\xi} \int_\xi^T \|S^{\frac12}(h_{\Delta t}(.,t) - h_{\Delta t}(.,t-\xi))\|^2_{L^2(\Omega)} dt \le C.$$

Let k be fixed ($1 \le k \le m$); for $\tau \in J_i$, we define the interval $Q = Q(\tau) = ((i-k)\Delta t, i\Delta t]$, and the characteristic function χ_Q.

Taking $v(x,t) = \chi_Q(t)\partial^{-k\Delta t}(h_{\Delta t}(x,\tau) - h_{D,\Delta t}(x,\tau)) \in l_{\Delta t}(V)$ in (3.10), we obtain

$$\begin{aligned}&\int_J (S\partial^{-\Delta t} h_{\Delta t}, \chi_Q(t)\partial^{-k\Delta t} h_{\Delta t}) dt \\ &= \int_J (S\partial^{-\Delta t} h_{\Delta t}, \chi_Q(t)\partial^{-k\Delta t} h_{D,\Delta t}) dt \\ &\quad - \int_J (K(x) T_s(h_{\Delta t}) \nabla h_{\Delta t}, \nabla \chi_Q(t) \partial^{-k\Delta t}(h_{\Delta t} - h_{D,\Delta t})) dt \\ &\quad + \int_J (K(x) T_s(h_{\Delta t}) \nabla f_{\Delta t}, \nabla \chi_Q(t) \partial^{-k\Delta t}(h_{\Delta t} - h_{D,\Delta t})) dt \\ &\quad - \varepsilon \int_J (\nabla h_{\Delta t}, \nabla \chi_Q(t) \partial^{-k\Delta t}(h_{\Delta t} - h_{D,\Delta t})) - \int_J (I_s, \chi_Q(t)\partial^{-k\Delta t}(h_{\Delta t} - h_{D,\Delta t})) dt\end{aligned}$$

applying the relation

$$\int_J \partial^{-\Delta t} h_{\Delta t} \chi_Q dt = k\Delta t \partial^{-k\Delta t} h_{\Delta t}(.,\tau),$$

and integrating again from $k\Delta t$ to T, we claim that we obtain

$$\begin{aligned}&k\Delta t \int_{k\Delta t}^T \|S^{\frac12} \partial^{-k\Delta t} h_{\Delta t}(.,\tau)\|^2_{L^2(\Omega)} d\tau \\ &\le C + k\Delta t \int_{k\Delta t}^T (S\partial^{-k\Delta t} h_{\Delta t}(.,\tau), \partial^{-k\Delta t} h_{D,\Delta t}(.,\tau)) d\tau.\end{aligned}$$

In fact, if we take for example the term

$$\int_{k\Delta t}^{T}\int_{J}(K(x)T_s(h_{\Delta t})\nabla h_{\Delta t}(x,t),\nabla\chi_Q(t)\partial^{-k\Delta t}h_{\Delta t}(x,\tau))dtd\tau = I_1+I_2,$$

where

$$I_1=\frac{1}{k\Delta t}\int_{k\Delta t}^{T}\int_{J}(K(x)T_s(h_{\Delta t})\chi_Q(t)\nabla h_{\Delta t}(x,t),\nabla h_{\Delta t}(x,\tau))dtd\tau$$

and

$$I_2=\frac{1}{k\Delta t}\int_{k\Delta t}^{T}\int_{J}(K(x)T_s(h_{\Delta t})\nabla h_{\Delta t}(x,t),\nabla\chi_Q(t)h_{\Delta t}(x,\tau-k\Delta t))dtd\tau.$$

For I_1, we have $I_1\le I_{1,1}+I_{1,2}$, with

$$\begin{aligned}
I_{1,1}&=\frac{1}{k\Delta t}\int_{k\Delta t}^{T}\int_{J}K(x)T_s(h_{\Delta t})\chi_Q(t)|\nabla h_{\Delta t}(x,t)|^2dtd\tau,\\
I_{1,2}&=\frac{1}{k\Delta t}\int_{k\Delta t}^{T}\int_{J}K(x)T_s(h_{\Delta t})\chi_Q(t)|\nabla h_{\Delta t}(x,\tau)|^2dtd\tau.
\end{aligned}$$

Hence

$$I_{1,1}\le C\frac{(\Delta t)^2}{k\Delta t}\sum_{i=k}^{i=m}\sum_{j=i-k+1}^{j=i}\|\nabla h^j(x)\|^2_{L^2(\Omega)}=C\frac{(\Delta t)^2}{k\Delta t}\sum_{i=k}^{i=m}\sum_{j=1}^{j=k}\|\nabla h^{j+i-k+1}(x)\|^2_{L^2(\Omega)}$$

$$I_{1,1}\le C\frac{\Delta t}{k\Delta t}\sum_{j=1}^{j=k}\sum_{i=k}^{i=m}\Delta t\|\nabla h^{j+i-k}(x)\|^2_{L^2(\Omega)}\le C\|\nabla h_{\Delta t}\|^2_{L^2(\Omega_T)}$$

then by (3.12), $I_{1,1}\le C$. For $I_{1,2}$, we have

$$\begin{aligned}
I_{1,2}&=\frac{1}{k\Delta t}\int_{k\Delta t}^{T}\int_{J}K(x)T_s(h_{\Delta t})\chi_Q(t)|\nabla h_{\Delta t}(x,\tau)|^2dt\,d\tau\\
I_{1,2}&=\frac{1}{k\Delta t}\int_{k\Delta t}^{T}\int_{(i-k)\Delta t}^{i\Delta t}K(x)T_s(h_{\Delta t})|\nabla h_{\Delta t}(x,\tau)|^2dt\,d\tau\\
I_{1,2}&\le C\frac{k\Delta t}{k\Delta t}\int_{k\Delta t}^{T}|\nabla h_{\Delta t}(x,\tau)|^2dt\,d\tau\le C.
\end{aligned}$$

Therefore, $I_1\le C$. Similarly we prove that I_2 and all the other diffusive terms are bounded. Moreover, as for (3.17), we obtain

$$\begin{aligned}
k\Delta t\int_{k\Delta t}^{T}(S\partial^{-k\Delta t}h_{\Delta t}(.,\tau),\partial^{-k\Delta t}h_{D,\Delta t}(.,\tau))d\tau &\le C\int_{k\Delta t}^{T}\|\partial^{-k\Delta t}h_{D,\Delta t}\|_{L^1(\Omega)}d\tau\\
&\le C\|\partial_t h_D\|_{L^1(\Omega_T)}.
\end{aligned}$$

Consequently, the estimation is valid and the strong convergence can be deduced. □

We are now ready to prove the main theorem.

Proof of Theorem 3.2. By the lemmas above, there exists a subsequence, also denoted by $(h_{\Delta t}, f_{\Delta t})$, and $(h, f) \in L^2(J, H^1(\Omega))^2$ such that

$$h_{\Delta t} - h_{D,\Delta t} \to h - h_D, \quad \text{weakly in } L^2(J, V) \tag{3.18}$$

$$f_{\Delta t} - f_{D,\Delta t} \to f - h_D, \quad \text{weakly in } L^2(J, V) \tag{3.19}$$

$$h_{\Delta t} \to h \quad \text{strongly in } L^2(\Omega_T) \tag{3.20}$$

$$T_s(h_{\Delta t}) \to T_s(h) \quad \text{strongly in } L^2(\Omega_T) \tag{3.21}$$

$$h_{\Delta t} \to h \quad \text{a. e. in } \Omega_T. \tag{3.22}$$

Next, for any $v \in L^2(J; V)$, $v_{\Delta t} \in l_{\Delta t}(V)$ for Δt sufficiently small, where $v_{\Delta t}(x,t) = \Delta t^{-1} \int_{J_i} v(x,\tau) d\tau$. Observe that

$$\int_J (S\partial^{-\Delta t} h_{\Delta t}, v) dt = \int_J (S\partial^{-\Delta t} h_{\Delta t}, v_{\Delta t}) dt$$

and

$$\|\nabla v_{\Delta t}\|_{L^2(\Omega_t)} \le \|\nabla v\|_{L^2(\Omega_t)}.$$

By taking $v_{\Delta t}$ as test function in (3.10) and using (3.12) we get

$$\int_J (S\partial^{-\Delta t} h_{\Delta t}, v) dt = \int_J (S\partial^{-\Delta t} h_{\Delta t}, v_{\Delta t}) dt \le C\|\nabla v_{\Delta t}\|_{L^2(\Omega_t)} \le C\|\nabla v\|_{L^2(\Omega_t)}.$$

Consequently, for a subsequence $S\partial^{-\Delta t} h_{\Delta t}$ converges weakly in $L^2(J, V')$, if $v \in \mathcal{D}(\Omega_T)$, we have

$$\begin{aligned}
\langle S\partial^{-\Delta t} h_{\Delta t}, v\rangle_{\mathcal{D}'(\Omega_T),\mathcal{D}(\Omega_T)} &= \int_J (S\partial^{-\Delta t} h_{\Delta t}, v) dt \\
&= -\int_0^{T-\Delta t} (S h_{\Delta t}, \partial^{\Delta t} v) dt \\
&\to -\int_J (Sh, \partial_t v) dt = \langle S\partial_t h, v\rangle_{\mathcal{D}'(\Omega_T),\mathcal{D}(\Omega_T)},
\end{aligned}$$

Therefore,

$$S\partial^{-\Delta t} h_{\Delta t} \rightharpoonup S\partial_t h \quad \text{weakly in } L^2(J, V'). \tag{3.23}$$

Combining (3.18)-(3.23), and since $\cup_{M=1}^{\infty} l_{\Delta t}(V)$ is dense in $L^2(J, V)$, we obtain (3.7) and (3.8).

On other hand, if $v \in \mathcal{C}^\infty(\Omega_T)$ with $v(x,T) = 0$, we find that

$$\int_J (S\partial^{-\Delta t} h_{\Delta t}, v) dt + \int_0^{T-\Delta t} (S[h_{\Delta t} - h_0], \partial^{\Delta t} v) dt = \frac{1}{\Delta t} \int_{T-\Delta t}^{T} (S[h_{\Delta t} - h_0], v) dt,$$

which yields (3.6). Finally by (3.9) and (3.22) we find (3.4) and thus the proof of the theorem is complete. □

Proof of Lemma 3.3. In this subsection, we allow h being outside $[H_1 + \delta, H_2]$. Here $T_s(h)$ is extended continuously and constantly outside $[H_1 + \delta, H_2]$. Lemma 3.3 is purely an elliptic result, and will follow from the next proposition. For notational convenience the subscript Δt is omitted below.

Proposition 3.6. *In addition to assumptions (A1)-(A6), suppose that* $0 < \eta_* \leq \eta_1(x) \in L^\infty(\Omega)$ *and* $\eta_2(x) \in L^\infty(\Omega)$ *such that* $\eta_1(x)(H_1+\delta) \leq \eta_2(x) \leq \eta_1(x)H_2$. *Then, the following problem has a weak solution* $(h, f) \in (V + h_D) \times (V + f_D)$*:*

$$H_1 + \delta \leq h(x,t) \leq H_2, \quad a.e.\ on\ \Omega_T. \tag{3.24}$$

$$\begin{aligned}(\eta_1 h, v) + (K(x)T_s(h)\nabla h, \nabla v) + \varepsilon(\nabla h, \nabla v)\\ -(K(x)T_s(h)\nabla f, \nabla v) = -(I_s, v) + (\eta_2, v), \quad \forall v \in V\end{aligned} \tag{3.25}$$

$$(K(x)T_a\nabla f, \nabla w) - (K(x)T_s(h)\nabla h, \nabla w) = (I_s + I_f, w)\ \forall w \in V. \tag{3.26}$$

Proof. Let $\{v_i\}_{i=1}^\infty$ be a base for V, we set $V_m = \operatorname{span}\{v_1, \dots, v_m\}$. With V_m replacing V in (3.25) and (3.26), we obtain a Galerkin method.

For $v^j = \sum_{i=1}^m \beta_i^j v_i$, $j = 1, 2$, we introduce the mapping $\Phi_m : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ by

$$\Phi_m \begin{pmatrix}\beta^1\\ \beta^2\end{pmatrix} = \begin{pmatrix}\hat\beta^1\\ \hat\beta^2\end{pmatrix},$$

where

$$\begin{aligned}\hat\beta_i^1 &= (\eta_1(v^1+h_D), v_i) + (K(x)T_s(v^1+h_D)\nabla(v^1+h_D), \nabla v_i) + \varepsilon(\nabla(v^1+h_D), \nabla v_i)\\ &\quad - (K(x)T_s((v^1+h_D))\nabla(v^2+f_D), \nabla v_i) - (I_s, v_i) - (\eta_2, v_i),\\ \hat\beta_i^2 &= (K(x)T_a\nabla(v^2+f_D), \nabla v_i) - (K(x)T_s(v^1+h_D)\nabla(v^1+h_D), \nabla v_i)\\ &\quad - (I_s + I_f, v_i).\end{aligned}$$

By assumptions (A1)-(A6), Φ_m is continuous. Also, it can be shown that, for any $\mu > 0$,

$$\begin{aligned}\Phi_m \begin{pmatrix}\beta^1\\ \beta^2\end{pmatrix} \cdot \begin{pmatrix}\beta^1\\ \beta^2\end{pmatrix} &\geq \frac{1}{2}(\eta_* - \mu)\|v^1\|_{L^2(\Omega)}^2 + \frac{1}{2}(\varepsilon - \mu C')\|v^1\|_V^2\\ &\quad + [\frac{K_*\delta}{2} - C'\frac{\mu}{2}]\|v^2\|_V^2 - C(\mu, h_D, f_D, I_f, I_s)\\ &\quad + \frac{1}{2}(K(x)T_s(v^1+h_D)\nabla(v^1-v^2), \nabla(v^1-v^2)).\end{aligned}$$

Therefore, for fixed μ small enough, $\Phi_m \begin{pmatrix}\beta^1\\ \beta^2\end{pmatrix} \cdot \begin{pmatrix}\beta^1\\ \beta^2\end{pmatrix}$ is strictly positive for $|\beta^1| + |\beta^2|$ sufficiently large. As result, Φ_m has a zero; i.e., there is a solution to the Galerkin approximation.

As in proof of Lemma 3.4 it can be seen that this Galerkin solutions h^m and f^m are uniformly bounded in $H^1(\Omega)$ (independently of m), so there exists a subsequence $h^m \rightharpoonup h$ and $f^m \rightharpoonup f$ weakly in $H^1(\Omega)$ with $h \in V + h_D$ and $f \in V + f_D$. Moreover, $h_m \to h$ strongly in $L^2(\Omega)$ and a.e. on Ω. Therefore (h, f) satisfy (3.25) and (3.26). Finally, a standard maximum principle argument on (3.25) (with $v = (h - H_2)^+$) can be applied to show that $h \leq H_2$. To show that $h \geq H_1 + \delta$ it suffices to set $v = (h - H_1 - \delta)^-$ in (3.25) and $w = \frac{(T_a - \delta)}{T_a} v$ in (3.26). Summing the two equations, using $(A4)$ and the fact that the extention of T_s is equal to $(T_a - \delta)$ on $\{x \in \mathbb{R} : x \leq H_1 + \delta\}$ we obtain the desired result. $\square$

4. Study of the degenerate problem

In this section, we obtain the convergence of the solutions of the regularized problem to a weak solution of the degenerate problem, obtaining hence the main result of this paper. We first define the weak formulation of the degenerate problem.

Definition 4.1. A pair of functions (h, f), is called a weak solution to the degenerate problem (2.2)-(2.4) if the following proprieties are fulfilled

$$H_1 + \delta \le h \le H_2, \quad \text{a.e. on } \Omega_T. \tag{4.1}$$

$$h \in L^2(J, V) + h_D, \quad S\frac{\partial h}{\partial t} \in L^2(J, V'), \quad f \in L^2(J, V) + f_D, \tag{4.2}$$

$$-\int_J \langle S\partial_t h, \varphi\rangle dt = \int_J (Sh, \frac{\partial \varphi}{\partial t}) + (S(x)h_0(x), \varphi(x, 0)) \quad \forall \varphi \in \mathcal{D}(\Omega \times [0, T[), \tag{4.3}$$

$$\begin{aligned}&\int_J \langle S\partial_t h, v\rangle dt + \int_J (K(x)\sqrt{T_s(h)}\nabla\phi(h), \nabla v)dt - \int_J (K(x)T_s(h)\nabla f, \nabla v)dt \\ &= -\int_J (I_s, v)dt \quad \forall v \in L^2(0, T, V),\end{aligned} \tag{4.4}$$

$$\begin{aligned}&\int_J (K(x)T_a\nabla f, \nabla w)dt - \int_J (K(x)\sqrt{T_s(h)}\nabla\phi(h), \nabla w)dt \\ &= \int_J (I_s + I_f, w)dt, \quad \forall w \in L^2(0, T, V),\end{aligned} \tag{4.5}$$

where $\phi(s) = \int_{H_1}^s \sqrt{T_s(\xi)}d\xi$ is introduced to absorb the degeneracy of the equations.

Theorem 4.2. *Under the assumptions (A1)-(A6), the system* (2.2)-(2.4) *has a weak solution in the sense of definition 4.1.*

The proof of this theorem is based on the following lemmas

Lemma 4.3. *Let $(h_\varepsilon, f_\varepsilon)$ a solution sequence to regularized problem. Then we have the following estimates:*

$$\begin{gathered}\|\nabla\phi(h_\varepsilon)\|^2_{L^2(\Omega_T)} \le C\,, \quad \varepsilon\|\nabla h_\varepsilon\|^2_{L^2(\Omega_T)} \le C\,, \\ \|\nabla f_\varepsilon\|^2_{L^2(\Omega_T)} \le C\,, \quad \|S\partial_t h_\varepsilon\|^2_{L^2(J,V')} \le C\,.\end{gathered}$$

Proof. In this section C is a generic constant independent of ε. Take $v = h_\varepsilon - h_D \in V$ in (3.7) and $w = f_\varepsilon - f_D \in V$ in (3.8) and summing the two equalities to have, for every $\mu > 0$,

$$\begin{aligned}&\frac{1}{2}\int_{\Omega_T} \varepsilon|\nabla h_\varepsilon|^2 + \frac{1}{2}\int_{\Omega_T} (K(x)T_f(h_\varepsilon) - \mu C)|\nabla f_\varepsilon|^2 \\ &+ \frac{1}{2}\int_{\Omega_T} K(x)T_s(h_\varepsilon)|\nabla(h_\varepsilon - f_\varepsilon)|^2 \\ &\le -\int_J \langle S\partial_t h_\varepsilon, h_\varepsilon - h_D\rangle + \frac{1}{2}\int_{\Omega_T} \varepsilon|\nabla h_D|^2 + \frac{1}{2}\int_{\Omega_T} K(x)T_f(h_\varepsilon)|\nabla f_D|^2 \\ &\quad + \frac{1}{2}\int_{\Omega_T} K(x)T_s(h_\varepsilon)|\nabla(h_D - f_D)|^2 + \int_{\Omega_T} |I_s(h_\varepsilon - h_D)| \\ &\quad + \frac{1}{2\mu}\int_{\Omega_T} |I_s + I_f|^2 + \int_{\Omega_T} |I_s + I_f||f_D|\end{aligned}$$

Since $H_1+\delta\le h_\varepsilon\le H_2$ and $H_1+\delta\le h_D\le H_2$, we have

$$T_f(h_\varepsilon)=h_\varepsilon-H_1\ge\delta,$$
$$\int_{\Omega_T}|I_s(h_\varepsilon-h_D)|\le C\|I_s\|_{L^1(\Omega_T)}\,.$$

Moreover, the following identity can be deduced as in [1, Lemma 1.5],

$$\begin{aligned}&\int_J\langle S\partial_t h_\varepsilon,h_\varepsilon-h_D\rangle\\&=\int_J(S\partial_t h_D,h_\varepsilon-h_D)+\frac12\|S(h_\varepsilon-h_D)(T)\|^2_{L^2(\Omega)}-\frac12\|S(h_\varepsilon-h_D)(0)\|^2_{L^2(\Omega)}\end{aligned}$$

also we have

$$\int_J(S\partial_t h_D,h_\varepsilon-h_D)\le C\|\partial_t h_D\|_{L^1(\Omega_T)}\,.$$

Therefore, for μ small enough,

$$\begin{gathered}\varepsilon\|\nabla h_\varepsilon\|^2_{L^2(\Omega_T)}\le C\,,\\\|\nabla f_\varepsilon\|^2_{L^2(\Omega_T)}\le C\,,\\\int_{\Omega_T}K(x)T_s(h_\varepsilon)|\nabla(h_\varepsilon-f_\varepsilon)|^2\le C\,.\end{gathered}$$

Since

$$\|\nabla\phi(h_\varepsilon)\|^2_{L^2(\Omega_T)}\le C\int_{\Omega_T}K(x)T_s(h_\varepsilon)|\nabla h_\varepsilon|^2\,,$$

we have

$$\begin{aligned}&\|\nabla\phi(h_\varepsilon)\|^2_{L^2(\Omega_T)}\\&\le 2C\int_{\Omega_T}K(x)T_s(h_\varepsilon)|\nabla(h_\varepsilon-f_\varepsilon)|^2+2C\int_{\Omega_T}K(x)T_s(h_\varepsilon)|\nabla f_\varepsilon|^2\le C.\end{aligned}$$

To show the last estimate, let $v\in L^2(J,V)$, we have

$$\int_J\langle S\partial_t h_\varepsilon,v\rangle dt\le\int_J(K(x)T_s(h_\varepsilon)\nabla(f_\varepsilon-h_\varepsilon),\nabla v)dt+\varepsilon\int_J(\nabla h_\varepsilon,\nabla v)dt+\int_J(I_s,v)dt,$$

Then

$$\int_J\langle S\partial_t h_\varepsilon,v\rangle dt\le C\|v\|_{L^2(J,V)}$$

which completes the proof. □

Lemma 4.4. *The sequence $(h_\varepsilon,f_\varepsilon)$, also satisfies the following inequality*

$$\int_\xi^T(S(h_\varepsilon(.,t)-h_\varepsilon(.,t-\xi)),\phi(h_\varepsilon(.,t))-\phi(h_\varepsilon(.,t-\xi)))dt\le C\xi\quad\forall\xi\in[0,T],\tag{4.6}$$

and we can extract a subsequence, also denoted $(h_\varepsilon,f_\varepsilon)$, such that, $(h_\varepsilon,\phi(h_\varepsilon))$ converges strongly to $(h,\phi(h))$ in $L^2(\Omega_T)$.

Proof. Let $\xi\in[0,T]$, and $v\in L^2(J,V)$. We have

$$\int_\xi^T(S(h_\varepsilon(.,t)-h_\varepsilon(.,t-\xi)),v)dt\le\int_\xi^T\|S(h_\varepsilon(.,t)-h_\varepsilon(.,t-\xi))\|_{V'}\|v\|_V dt,$$

moreover we know that (see [7, p. 155]), for all $\xi \in [0, T]$,

$$\frac{1}{\xi^2} \int_{\xi}^{T} \|S(h_\varepsilon(.,t) - h_\varepsilon(.,t-\xi))\|_{V'}^2 dt \le \int_0^T \|\partial_t S h_\varepsilon\|_{V'}^2 dt \,.$$

Then

$$\int_{\xi}^{T} (S(h_\varepsilon(.,t) - h_\varepsilon(.,t-\xi)), v) dt \le \xi \|v\|_{L^2(J,V)} \|S\partial_t h_\varepsilon\|_{L^2(J,V')} \,;$$

therefore, taking $v = \phi(h_\varepsilon(.,t)) - \phi(h_\varepsilon(.,t-\xi)$ and by the previous lemma we have the estimate (4.6).

On the other hand we have $T_s(\xi) = H_2 - \xi$. Then ϕ is continuous strictly decreasing function on $[H_1, H_2]$, and $\mathcal{C}^1$ on $]H_1, H_2[$. Consequently ϕ^{-1} is lipschitz function on $[H_1, H_2]$. Hence, since $H_1 + \delta \le h_\varepsilon \le H_2$, we obtain

$$\int_{\xi}^{T} S(\phi^{-1} \circ \phi(h_\varepsilon(.,t)) - \phi^{-1} \circ \phi(h_\varepsilon(.,t-\xi)), \phi(h_\varepsilon(.,t)) - \phi(h_\varepsilon(.,t-\xi))) dt \le C\xi$$

and

$$\int_{\xi}^{T} (S(x)(\phi(h_\varepsilon(.,t)) - \phi(h_\varepsilon(.,t-\xi)), \phi(h_\varepsilon(.,t)) - \phi(h_\varepsilon(.,t-\xi))) dt \le C\xi$$

Moreover,

$$\|\nabla \phi(h_\varepsilon)\|_{L^2(\Omega_T)}^2 \le C \,.$$

Therefore, as in [9, Lemma 2.6], we deduce that $\phi(h_\varepsilon)$ converges strongly in $L^2(\Omega_T)$ to $\phi(h)$. □

Proof of Theorem 4.2. By Lebesque's theorem, lemma 4.3 and lemma 4.4 we deduce that there exist $(h, f) \in (L^2(J, V) + h_D) \times (L^2(J, V) + f_D)$ and a subsequence $(h_\varepsilon, f_\varepsilon)$ such that

$$\begin{gathered} h_\varepsilon \to h \quad \text{strongly in } L^p(\Omega_T) \quad \forall p \in [1, \infty[. \\ \phi(h_\varepsilon) \to \phi(h) \quad \text{strongly in } L^p(\Omega_T)\ \forall p \in [1, \infty[\,, \\ S\partial_t h_\varepsilon \to S\partial_t h \quad \text{weakly in } L^2(J, V')\,, \\ f_\varepsilon - f_D \to f - f_D \quad \text{weakly in } L^2(J, V), \\ \phi(h_\varepsilon) - \phi(h_D) \to \phi(h) - \phi(h_D) \quad \text{weakly in } L^2(J, V), \\ \sqrt{\varepsilon}(h_\varepsilon - h_D) \to 0 \quad \text{weakly in } L^2(J, V), \\ h_\varepsilon \to h \quad \text{a. e. in } \Omega_T. \end{gathered}$$

Since $\phi(h_D) \in L^2(J, H^1(\Omega))$, as ε approaches 0 in (3.7) and in (3.8), we obtain (4.4) and (4.5). To show (4.1) and (4.3) it suffices to proceed as in theorem 3.2. □

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