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COEXISTENCE AND STABILITY OF SOLUTIONS FOR A CLASS OF REACTION-DIFFUSION SYSTEMS

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ABSTRACT. In this paper we consider the situation of two species of predators competing for one species of prey. We use comparison principles to study the global existence, the existence of non-trivial steady states and their stability.

1. INTRODUCTION

It is well-known that the study of coexistence problem of competing species is one of the main topics in mathematical ecology. The object of this paper is to study the problem of coexistence for three interacting species, among which two species of predators compete for one species of prey. We assume that the two competing species have different diffusion rates $d_1 < d_2$. The model is

$$
\begin{gathered}
\frac{\partial u}{\partial t} = d_0 \triangle u + F_0(u, v_1, v_2), \quad \text{on } \Omega \times \mathbb{R}_+, \\
\frac{\partial v_1}{\partial t} = d_1 \triangle v_1 + F_1(u, v_1, v_2), \quad \text{on } \Omega \times \mathbb{R}_+, \\
\frac{\partial v_2}{\partial t} = d_2 \triangle v_2 + F_2(u, v_1, v_2), \quad \text{on } \Omega \times \mathbb{R}_+, \\
\frac{\partial u}{\partial \nu}(x,t) + r_0(x) u(x,t) = u^0(x), \quad x \in \partial\Omega,\ t > 0, \\
\frac{\partial v_i}{\partial \nu}(x,t) + r_i(x) v_i(x,t) = 0, \quad i = 1, 2,\ x \in \partial\Omega,\ t > 0, \\
u(x,0) = u_0(x), v_i(x,0) = v_{i0}(x), \quad i = 1, 2, \text{ in } \Omega,
\end{gathered}
\tag{1.1}
$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. $\triangle$ is the Laplacian. ν denotes the outer unit normal to $\partial\Omega$. u is the population density of the prey, and v_1, v_2 are the population densities of two competing predator species. $d_i > 0$ are the diffusion rates with $d_1 < d_2$. $r_0(x) > 0$, $r_2(x) \ge r_1(x) \ge 0 \not\equiv 0$, $u^0(x) \ge 0 \not\equiv 0$, $u_0(x) \ge 0$, $v_{i0}(x) \ge 0$. F_i are given by

$$
\begin{aligned}
F_0(u, v_1, v_2) &= -f_1(u) v_1 - f_2(u) v_2, \\
F_1(u, v_1, v_2) &= v_1 (f_1(u) - v_1 - v_2), \\
F_2(u, v_1, v_2) &= v_2 (f_2(u) - v_1 - v_2).
\end{aligned}
\tag{1.2}
$$

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Here $f_i(u)$ is the consumption rate of the prey per predator. The forms of F_1 and F_2 represent that, at the constant level of the prey u, the predators have logistic growth. A similar model has been investigated by Wang and Wu in [16] by using bifurcation theory when the predators have a Malthusian (or exponential) growth. The fact that predator species have different diffusion rates makes it hard to study this model. If assuming equal diffusion rates, the model can be simplified to a much simpler model (e.g see [8]). We assume that

(i) $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ and $f_i(0) = 0$;
(ii) f_i is continuously differentiable and $f_i' \geq 0 \not\equiv 0$.

A typical example of f_i is the monotone Monod function, $f_i(u) = m_i u/(a_i + u)$ (see [8], [10]), where a_i and m_i are positive constants. m_i is the maximal growth rate and a_i is the Michaelis-Menten (or half-saturation) constant.

Since u and v_i are population densities, so only non-negative solutions are of physical interest. Observe that $F_0(0, v_1, v_2) = 0$ and $F_i(u, v_1, v_2) = 0$ if $v_i = 0$ for $i = 1, 2$, therefore

$$\mathbb{R}^3_+ = \{(u, v_1, v_2)|\ u \geq 0,\ v_i \geq 0,\ i = 1, 2\}$$

is an invariant region of (1.1) (see [14]). Therefore in this paper we consider only nonnegative solutions of (1.1) without further explanation.

This paper is organized as follows: In Section 2, we will prove the global existence of solutions. In Section 3, we consider one predator population case in detail. In Section 4, we will study the existence of steady states. In Section 5, we study the local stability of equilibrium solutions.

2. Existence of a global solution

By standard existence theory, e.g. see [1], [2] and [3], (1.1) has a unique nonnegative smooth local solution $U(x,t) = (u(x,t), v_1(x,t), v_2(x,t))$ existing for $0 \leq t < T_{\max}$ and it is well-known that local existence together with L^∞ a priori bounds ensure the global existence of classical solutions.

Lemma 2.1. *The boundary value problem*

$$\begin{gathered} \triangle S = 0, \quad \text{on } \Omega, \\ \frac{\partial S}{\partial \nu}(x,t) + r_0(x) S(x) = u^0(x), \quad x \in \partial\Omega, \end{gathered} \tag{2.1}$$

has a unique strictly positive solution $S(x) > 0$ for all $x \in \overline{\Omega}$.

For the proof of this Lemma see [10] and the references therein.

Remark 2.2. The importance of this Lemma lies in that:
(i) It implies that (1.1) has an equilibrium solution $(u, v_1, v_2) = (S(x), 0, 0)$. This is so-called washout equilibrium solution. In practice, attaining this equilibrium is undesirable.
(ii) It provides an a priori bound for u, which ensures the existence of the global solution.

Lemma 2.3. *For any solution of the form $(0, v_1(x,t), v_2(x,t))$ of (1.1), we have $v_i(x,t) \to 0$ as $t \to \infty$. That is, if there is no prey, then the predators will be extinct.*

Proof. From our assumptions about f_i, if $u = 0$, the equation for $v_i (i = 1, 2)$ becomes

$$\frac{\partial v_i}{\partial t} = d_i \triangle v_i + v_i(-\sum_{j=1}^{2} v_j) \le d_i \triangle v_i - v_i^2.$$

Let $A_i = \max_{x \in \overline{\Omega}} v_{i0}(x)$ and compare v_i with the solution of the initial value problem of ODE:

$$\begin{gathered}\frac{dw_i}{dt} = -w_i^2, \quad t > 0,\\ w(0) = A_i,\end{gathered}$$

we have

$$0 \le v_i(x,t) \le w_i(t).$$

But $w_i(t) \to 0$ as $t \to \infty$, therefore, $v_i(x,t) \to 0$ as $t \to \infty$. □

By virtue of $S(x)$ in Lemma 2.1 and comparison principle (e.g. see [12], [13], [17]), it is easy to prove the following global existence and uniqueness theorem. We omit the proof here.

Theorem 2.4. *For any smooth nonnegative functions $u^0(x)$, $u_0(x)$ and $v_{i0}(x)$,* (1.1) *has a unique smooth bounded global solution.*

This theorem implies that the solutions of (1.1) generate a semidynamical system on $C_+ \times C_+ \times C_+$, where C_+ is the set of nonnegative, continuous functions on $\bar{\Omega}$ with the usual supremum norm. This semidynamical system is denoted by $\Phi(t, x_0)$, where $t \ge 0$ and x_0 represents the triple of initial conditions given by (1.1).

3. One Predator Case

In this section, we consider a special case when there is only one predator population, or equivalently, $v_i \equiv 0$ for $i = 1$ or $i = 2$. Without loss of generality, we assume that $v_2 \equiv 0$ and write v_1 as v and f_1 as f respectively, then (1.1) becomes

$$\begin{gathered}\frac{\partial u}{\partial t} = d_0 \triangle u - f(u)v, \quad \text{on } \Omega \times \mathbb{R}_+,\\ \frac{\partial v}{\partial t} = d_1 \triangle v + v(f(u) - v), \quad \text{on } \Omega \times \mathbb{R}_+,\\ \frac{\partial u}{\partial \nu}(x,t) + r_0(x) u(x,t) = u^0(x), \quad x \in \partial\Omega,\ t > 0,\\ \frac{\partial v}{\partial \nu} + r_1(x) v(x,t) = 0, \quad x \in \partial\Omega,\ t > 0,\\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad \text{in } \Omega.\end{gathered} \tag{3.1}$$

3.1. **Steady States.** First, from Lemma 2.1, (3.1) has a steady state solution $U_w = (u,v) = (S(x), 0)$ with $S(x) > 0$. The following theorem says that (3.1) has a positive steady state solution $U_p = (u,v) = (\tilde{u}(x), \tilde{v}(x))$ with $\tilde{u}(x) > 0$ and $\tilde{v}(x) > 0$.

Theorem 3.1. (3.1) *has a positive steady state solution $(u,v) = (\tilde{u}(x), \tilde{v}(x))$ with $\tilde{u}(x) > 0$ and $\tilde{v}(x) > 0$ provided that $\int_{\partial\Omega} r_1(x)dx$ is small enough.*

To prove this Theorem, we need some preparations. Consider the eigenvalue problem

$$\begin{aligned} d\triangle\phi + q(x)\phi = \lambda\phi, \quad x \in \Omega,\\ \frac{\partial\phi}{\partial\nu} + r(x)\phi = 0, \quad x \in \partial\Omega, \end{aligned} \tag{3.2}$$

where $d > 0$, $q(x) \in C^{2+\alpha}(\bar{\Omega})$ for some $\alpha > 0$. It is well-known that there is a unique eigenvalue $\lambda_1 = \lambda(q, d, r)$, called the *'principal eigenvalue'*, such that the associated 'principal eigenfunction' (unique up to a multiplicative constant) is strictly positive. Furthermore, we have the following Lemma.

Lemma 3.2. *The principal eigenvalue $\lambda(q, d, r)$ of (3.2) is a continuous non-increasing function of d, and is strictly decreasing if $q(x)$ is not a constant. Furthermore, the following hold:*

(a) $\lambda(q, d, r) \uparrow Q = \max_{\bar{\Omega}} q(x)$ *as* $d \to 0$;
(b) $\lambda(q, d, r) \downarrow \omega = \frac{1}{|\Omega|}\int_\Omega q(x)dx - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} r(x)dx$ *as* $d \to \infty$;
(c) *If* $q_1(x) \geq q_2(x)$ *for* $x \in \Omega$, *then* $\lambda(q_1, d, r) \geq \lambda(q_2, d, r)$ *with strict inequality if* $q_1(x) \not\equiv q_2(x)$;
(d) *If* $r_1(x) \leq r_2(x)$ *for* $x \in \partial\Omega$, *then* $\lambda(q, d, r_1) \geq \lambda(q, d, r_2)$.

For the proof of the above Lemma, see [4] and the references therein.

Remark 3.3. From this Lemma we can see that if $\int_\Omega q(x)dx > 0$ and $\int_{\partial\Omega} r(x)dx$ is small such that $\omega > 0$, then for any $d > 0$, the principal eigenvalue $\lambda(q, d, r)$ of (3.2) is positive. In particular, if $q(x) \geq 0 \not\equiv 0$ and $r(x) \equiv 0$, i.e. for Neumann boundary condition, the principal eigenvalue of (3.2) is always positive.

By using this Lemma, we can prove the following Lemma.

Lemma 3.4. *If the principal eigenvalue $\lambda(q, d, r)$ of (3.2) is positive, then the boundary-value problem*

$$\begin{aligned} d\triangle u + u(q(x) - u) = 0, \quad x \in \Omega,\\ \frac{\partial u}{\partial\nu} + r(x)u = 0, \quad x \in \partial\Omega, \end{aligned} \tag{3.3}$$

has a unique, strictly positive solution.

Proof. Let $\psi(x) > 0$ be the principal eigenfunction of (3.2) corresponding to λ_1, and let $\underline{u}(x) = \delta\psi(x)$, where $\delta > 0$ is a small number to be determined, then for $\delta > 0$ small enough,

$$\begin{gathered} d\triangle\underline{u} + \underline{u}(q(x) - \underline{u}) = \lambda_1\delta\psi - \delta^2\psi^2 > 0, \quad x \in \Omega,\\ \frac{\partial\underline{u}}{\partial\nu} + r(x)\underline{u} = \delta(\frac{\partial\psi}{\partial\nu} + r(x)\psi) = 0, \quad x \in \partial\Omega. \end{gathered}$$

Therefore, $\underline{u}(x)$ is a sub-solution of (3.3). It is easily seen that $\bar{u}(x) = Q = \max_{\bar{\Omega}} q(x)$ is a sup-solution of (3.3). Hence (3.3) has a strictly positive solution $u(x)$ satisfying $0 < \delta\psi \leq u(x) \leq Q$.

Now we prove the uniqueness. Suppose that u_1 and u_2 both are positive solutions of (3.3). Let $w = \frac{u_1}{u_2} > 0$, then w satisfies

$$\begin{gathered} d\triangle w + \frac{2d\nabla u_2}{u_2}\nabla w + wu_2(1 - w) = 0, \quad x \in \Omega,\\ \partial_\nu w|_{\partial\Omega} = 0. \end{gathered}$$

Then using the maximum principle, we have $w \equiv 1$. This completes the proof of the Lemma. $\square$

Proof of Theorem 3.1. To prove the Theorem, we need to prove the existence of positive solutions of the boundary-value problem

$$\begin{gathered} d_0 \triangle u - f(u)v = 0, \quad x \in \Omega, \\ d_1 \triangle v + v(f(u) - v) = 0, \quad x \in \Omega, \\ \frac{\partial u}{\partial \nu} + r_0(x)u = u^0(x), \quad x \in \partial\Omega, \\ \frac{\partial v}{\partial \nu} + r_1(x)v = 0, \quad x \in \partial\Omega. \end{gathered} \tag{3.4}$$

For notational convenience, we write (3.4) as

$$\begin{gathered} -d_0 \triangle u = g_1(u, v), \quad x \in \Omega, \\ -d_1 \triangle v = g_2(u, v), \quad x \in \Omega, \\ \frac{\partial u}{\partial \nu} + r_0(x)u = u^0(x), \quad x \in \partial\Omega, \\ \frac{\partial v}{\partial \nu} + r_1(x)v = 0, \quad x \in \partial\Omega, \end{gathered} \tag{3.5}$$

where $g_1(u, v) = -f(u)v$, $g_2(u, v) = v(f(u) - v)$. Then we can see that for $u \geq 0$ and $v \geq 0$,

$$\frac{\partial g_1}{\partial v} = -f(u) \leq 0, \quad \frac{\partial g_2}{\partial u} = vf'(u) \geq 0.$$

That is g_1 is quasi-monotonic decreasing and g_2 is quasi-monotonic increasing. Therefore, (3.5) is a so-called mixed quasi-monotonic system. For such a system, we have the following definition of upper and lower solutions (e.g. see [12] and [17]).

Definition: If $U(x) = (\bar{u}(x), \bar{v}(x))$ and $V(x) = (\underline{u}(x), \underline{v}(x))$ satisfy

$$\begin{gathered} \frac{\partial \bar{u}}{\partial \nu} + r_0(x)\bar{u} \geq u^0(x) \geq \frac{\partial \underline{u}}{\partial \nu} + r_0(x)\underline{u}, \quad \text{on } \partial\Omega, \\ \frac{\partial \bar{v}}{\partial \nu} + r_1(x)\bar{v} \geq 0 \geq \frac{\partial \underline{v}}{\partial \nu} + r_1(x)\underline{v}, \quad \text{on } \partial\Omega, \\ -d_0 \triangle \bar{u} - g_1(\bar{u}, \underline{v}) \geq 0 \geq -d_0 \triangle \underline{u} - g_1(\underline{u}, \bar{v}), \quad \text{in } \Omega, \\ -d_1 \triangle \bar{v} - g_2(\bar{u}, \bar{v}) \geq 0 \geq -d_1 \triangle \underline{v} - g_2(\underline{u}, \underline{v}), \quad \text{in } \Omega, \end{gathered}$$

then $(U(x), V(x))$ is said to be a pair of upper and lower solutions of (3.5).

Then we have the following result.

Theorem 3.5. *If* (3.5) *has a pair of upper and lower solutions* $(U(x), V(x))$ *such that* $V(x) \leq U(x)$*, then* (3.5) *has at least one solution* $(u(x), v(x))$ *satisfying*

$$V(x) \leq (u(x), v(x)) \leq U(x).$$

The proof of this theorem can be found in [17].

Now we construct a pair of upper and lower solutions $U(x) = (\bar{u}(x), \bar{v}(x))$ and $V(x) = (\underline{u}(x), \underline{v}(x))$ as follows: First observe that, by the definition, we need $U(x) = (\bar{u}(x), \bar{v}(x))$ and $V(x) = (\underline{u}(x), \underline{v}(x))$ satisfy

$$-d_0 \triangle \bar{u} + \underline{v} f(\bar{u}) \geq 0 \geq -d_0 \triangle \underline{u} + \bar{v} f(\underline{u}), \tag{3.6}$$

$$-d_1 \triangle \bar{v} - \bar{v}(f(\bar{u}) - \bar{v}) \geq 0 \geq -d_1 \triangle \underline{v} - \underline{v}(f(\underline{u}) - \underline{v}). \tag{3.7}$$

Let $\bar{u}(x) = S(x)$ be the unique positive solution of (2.1), then, for any $\underline{v}(x) \geq 0$, the left-hand side of (3.6) is satisfied. We take $\bar{v}(x) = M_1$ to be a positive constant such that $M_1 \geq f(\max_{\bar{\Omega}} S(x))$, then the left-hand side of (3.7) is satisfied. With $\bar{v}(x) = M_1$, the right-hand side of (3.6) becomes

$$-d_0 \triangle \underline{u} + M_1 f(\underline{u}) \leq 0.$$

We take $\underline{u} = \underline{u}(x) > 0$ to be the positive solution of the boundary-value problem

$$\begin{gathered} -d_0 \triangle \underline{u} + M_1 f(\underline{u}) = 0, \quad x \in \Omega, \\ \frac{\partial \underline{u}}{\partial \nu} + r_0(x)\underline{u} = u^0(x), \quad x \in \partial\Omega. \end{gathered} \tag{3.8}$$

Now we prove that (3.8) has a positive solution. In fact, since $f(0) = 0$, 0 is a lower solution of (3.8). Since $f \geq 0$, any constant K satisfying

$$K \geq \frac{1}{\gamma} \max_{\bar{\Omega}} u^0(x),$$

where $\gamma = \min_{\bar{\Omega}} r_0(x) > 0$, is an upper solution of (3.8). Therefore (3.8) has a solution $\underline{u}(x)$ satisfying $0 \leq \underline{u}(x) \leq K$. By strong maximum principle, we have

$$\underline{u}(x) > 0, \quad \text{for} \quad x \in \bar{\Omega}.$$

We claim that $\bar{u}(x) \geq \underline{u}(x)$. In fact, let $w = \bar{u}(x) - \underline{u}(x)$, then w satisfies

$$\begin{gathered} -d_0 \triangle w = M_1 f(\underline{u}) \geq 0, \quad x \in \Omega, \\ \frac{\partial w}{\partial \nu} + r_0 w = 0, \quad x \in \partial\Omega. \end{gathered}$$

By the maximum principle

$$\min_{\bar{\Omega}} w = \min_{\partial\Omega} w.$$

Now we prove that $\min_{\partial\Omega} w \geq 0$. Indeed, if $w(x_0) = \min_{\partial\Omega} w < 0$, then, by Hopf's Lemma, at x_0, we have $\frac{\partial w}{\partial \nu} < 0$. But from the boundary condition, we have $\frac{\partial w}{\partial \nu}|_{x_0} = -r_0 w(x_0) > 0$. This is a contradiction. Therefore $w \geq 0$. That is $\bar{u}(x) \geq \underline{u}(x)$. With $\underline{u} = \underline{u}(x) > 0$, $f(\underline{u}) \geq 0 \not\equiv 0$. Therefore, $\int_\Omega f(\underline{u}(x))dx > 0$. By Lemma 3.4, if $\int_{\partial\Omega} r_1(x)dx$ is small enough, then the boundary-value problem

$$\begin{gathered} -d_1 \triangle \underline{v} - \underline{v}(f(\underline{u}) - \underline{v}) = 0, \quad x \in \Omega, \\ \frac{\partial \underline{v}}{\partial \nu} + r_1(x)\underline{v} = 0, \quad x \in \partial\Omega, \end{gathered} \tag{3.9}$$

has a unique positive solution $\underline{v} = \underline{v}(x) > 0$.

Now we claim that $\underline{v} \leq M_1 = \bar{v}$. In fact, from the proof of Lemma 3.4, we can see that the positive solution of (3.9) is bounded from above by any constant that is greater than or equal to $f(\max_{\bar{\Omega}} \underline{u}(x))$. Since $\underline{u}(x) \leq \bar{u}(x)$ and $f' \geq 0$, we have

$$\underline{v}(x) \leq f(\max_{\bar{\Omega}} \underline{u}(x)) \leq f(\max_{\bar{\Omega}} \bar{u}(x)) \leq M_1.$$

Thus we have constructed a pair of upper and lower solutions $U(x) = (\bar{u}(x), \bar{v}(x))$ and $V(x) = (\underline{u}(x), \underline{v}(x))$ of (3.5) satisfying $V(x) \leq U(x)$, then from Theorem A we know that (3.5) has a solution $(\tilde{u}(x), \tilde{v}(x))$ satisfying

$$V(x) \leq (\tilde{u}(x), \tilde{v}(x)) \leq U(x).$$

□

For the rest of this article, we assume that $\int_{\partial\Omega} r_1(x)dx$ is small enough such that the related eigenvalue problem with Robin boundary condition has a positive principal eigenvalue, without further explanation.

Now we prove some properties of positive solutions of (3.5).

Proposition 3.6. *Suppose that (u_1, v_1) and (u_2, v_2) are two positive solutions of (3.4). If $u_1 \geq u_2$, then $v_1 \geq v_2$.*

Before proving the above proposition, we cite the following Lemma whose proof can be found in [4].

Lemma 3.7. *Consider the initial boundary value problem*

$$\begin{gathered}
u_t = d\triangle u + u(q(x) - u), \quad x \in \Omega,\ t > 0,\\
\frac{\partial u}{\partial \nu} + r_1(x)u = 0, \quad x \in \partial\Omega,\ t > 0,\\
u(x, 0) = u_0(x).
\end{gathered} \tag{3.10}$$

If $\lambda(q, d, r_1) \leq 0$, then 0 is the global attractor for positive solutions of (3.10). If $\lambda(q, d, r_1) > 0$, then the unique, strictly positive steady-state solution is a global attractor for non-trivial positive solutions of (3.10), the convergence in both cases being in $\|\cdot\|_\infty$.

Proof of Proposition 3.6. We consider the following initial-boundary-value problems:

$$\begin{gathered}
V_{1t} = d_1\triangle V_1 + V_1(f(u_1(x)) - V_1), \quad x \in \Omega,\ t > 0,\\
\frac{\partial V_1}{\partial \nu} + r_1(x)V_1 = 0, \quad x \in \partial\Omega,\ t > 0,\\
V_1(x, 0) = v_0(x),
\end{gathered} \tag{3.11}$$

and

$$\begin{gathered}
V_{2t} = d_1\triangle V_2 + V_2(f(u_2(x)) - V_2), \quad x \in \Omega,\ t > 0,\\
\frac{\partial V_2}{\partial \nu} + r_1(x)V_2 = 0, \quad x \in \partial\Omega,\ t > 0,\\
V_1(x, 0) = v_0(x).
\end{gathered} \tag{3.12}$$

From the positivity of $f(u_i)$ and Lemma 3.7 we know that $V_1(x, t) \to v_1(x)$ as $t \to \infty$ and $V_2(x, t) \to v_2(x)$ as $t \to \infty$. $u_1 \geq u_2$ implies that $f(u_1) \geq f(u_2)$. Therefore, from (3.11), we have

$$V_{1t} \geq d_1\triangle V_1 + V_1(f(u_2(x)) - V_1).$$

Thus $V_1(x, t)$ is an upper solution of (3.12). So we have $V_1(x, t) \geq V_2(x, t)$. Therefore, $v_1 \geq v_2$. □

Proposition 3.8. *Suppose that (u_1, v_1) and (u_2, v_2) are two positive solutions of (3.4). If $u_1 \geq u_2$, then $u_1 \equiv u_2$, $v_1 \equiv v_2$.*

Proof. Let $w = u_1 - u_2$, then we have $w \geq 0$. u_1 and u_2 satisfy

$$d_0\triangle u_1 = f(u_1)v_1 \geq f(u_2)v_2 = d_0\triangle u_2.$$

Therefore, w satisfies

$$\begin{gathered}
-d_0\triangle w \leq 0, \quad x \in \Omega,\\
\frac{\partial w}{\partial \nu} + r_0 w = 0, \quad x \in \partial\Omega.
\end{gathered}$$

Hence, if $w \not\equiv 0$, by the maximum principle,

$$\max_{\bar{\Omega}} w = \max_{\partial\Omega} w > 0.$$

Assume that $x_0 \in \partial\Omega$ such that

$$w(x_0) = \max_{\partial\Omega} w > 0,$$

then, by Hopf's Lemma, at x_0, we have $\frac{\partial w}{\partial \nu} > 0$. This contradicts to the boundary condition. Therefore we have $w \equiv 0$, i.e. $u_1 \equiv u_2$. Then we must have $v_1 \equiv v_2$. $\square$

Proposition 3.9. *Suppose that* (u_1, v_1) *and* (u_2, v_2) *are two positive solutions of* (3.4). *If* $f(u_1) - f(u_2) \geq v_1 - v_2$, *then* $u_1 \equiv u_2$, $v_1 \equiv v_2$.

Proof. From Proposition 3.8, we need only to prove that $u_1 \geq u_2$. In fact, due to the positivity of v_i, v_i can be looked as the principal eigenfunction of the eigenvalue problem

$$\begin{gathered} d_1 \triangle \phi + q_i(x)\phi = \lambda\phi, \quad x \in \Omega, \\ \frac{\partial \phi}{\partial \nu} + r_1(x)\phi = 0, \quad x \in \partial\Omega, \end{gathered}$$

with $q_i(x) = f(u_i) - v_i$, and then associated with the principal eigenvalue $\lambda = \lambda(q_i(x), d_1, r_1) = 0$, $i = 1, 2$. Since $f(u_1) - f(u_2) \geq v_1 - v_2$, we have

$$f(u_1) - v_1 \geq f(u_2) - v_2.$$

Therefore, by Lemma 3.2, we have $f(u_1) - v_1 \equiv f(u_2) - v_2$, that is $f(u_1) - f(u_2) \equiv v_1 - v_2$. Let $w = u_1 - u_2$, then w satisfies

$$\begin{gathered} -d_0 \triangle w + f'(\theta)(v_1 + f(u_2))w = 0, \quad x \in \Omega \\ \frac{\partial w}{\partial \nu} + r_0 w = 0, \quad x \in \partial\Omega. \end{gathered}$$

Therefore, if $w < 0$ somewhere, by maximum principle,

$$\min_{\bar{\Omega}} w = \min_{\partial\Omega} w < 0.$$

Assume that $x_0 \in \partial\Omega$ such that

$$w(x_0) = \min_{\partial\Omega} w < 0,$$

then, by Hopf's Lemma, at x_0, we have $\frac{\partial w}{\partial \nu} < 0$. This contradicts to the boundary condition. Therefore, $w \geq 0$, that is $u_1 \geq u_2$. $\square$

3.2. **Stability of Steady States.**

Theorem 3.10. $U_w = (S(x), 0)$ *is unstable.*

Proof. The linearized system of (3.1) around U_w is

$$\begin{gathered} w_{0t} = d_0 \triangle w_0 - f(S)w_1, \quad x \in \Omega, \\ w_{1t} = d_1 \triangle w_1 + f(S)w_1, \quad x \in \Omega, \\ \frac{\partial w_0}{\partial \nu} + r_0(x)w_0 = u^0(x), \quad x \in \partial\Omega, \\ \frac{\partial w_1}{\partial \nu} + r_1(x)w_1 = 0, \quad x \in \partial\Omega. \end{gathered}$$

Now we study the eigenvalue problem

$$\begin{gathered} d_0\triangle w_0 - f(S)w_1 = \eta w_0, \quad x \in \Omega, \\ d_1\triangle w_1 + f(S)w_1 = \eta w_1, \quad x \in \Omega, \\ \frac{\partial w_0}{\partial \nu} + r_0(x)w_0 = 0, \quad x \in \partial\Omega, \\ \frac{\partial w_1}{\partial \nu} + r_1(x)w_1 = 0, \quad x \in \partial\Omega. \end{gathered} \tag{3.13}$$

To prove the Theorem, we need to prove that the largest eigenvalue of (3.13) is positive. Let η be an eigenvalue of (3.13) with eigenfunction (w_0, w_1). If $w_1 \not\equiv 0$, then η is an eigenvalue of $d_1\triangle + f(S)$ with homogeneous Robin boundary condition. Therefore, η must be real. If $w_1 \equiv 0$, then $w_0 \not\equiv 0$. So η is an eigenvalue of $d_0\triangle$ with homogeneous Robin boundary condition. Therefore, η is also real. So we know that all eigenvalues of (3.13) are real. Let η_1 be the largest eigenvalue of (3.13). Since $f(S) > 0$, the principal eigenvalue λ_1 of

$$\begin{gathered} d_1\triangle w_1 + f(S)w_1 = \lambda w_1, \quad x \in \Omega, \\ \frac{\partial w_1}{\partial \nu} + r_1(x)w_1 = 0, \quad x \in \partial\Omega, \end{gathered}$$

is positive and the associated eigenfunction $\tilde{w}_1 > 0$. We claim that λ_1 is also an eigenvalue of (3.13). In fact, let $\tilde{w}_0$ be the solution of the linear boundary-value problem

$$\begin{gathered} d_0\triangle w_0 - f(S)\tilde{w}_1 = \lambda_1 w_0, \quad x \in \Omega, \\ \frac{\partial w_0}{\partial \nu} + r_0(x)w_0 = 0, \quad x \in \partial\Omega, \end{gathered}$$

then $(w_0, w_1) = (\tilde{w}_0, \tilde{w}_1)$ satisfies (3.13) with $\eta = \lambda_1$. So $\lambda_1 > 0$ is an eigenvalue of (3.13). Therefore, $\eta_1 \ge \lambda_1 > 0$. Hence U_w is unstable. $\square$

Lemma 3.11. *Suppose that (u, v) is a positive solution of* (3.4)*, then we have*

$$\begin{gathered} 0 < u(x) \le S(x) \le \hat{S}, \\ 0 < v(x) \le \hat{v} \le \hat{V}, \end{gathered}$$

where $\hat{S} = \max_{\bar{\Omega}} S(x)$, and $\hat{V} = \max_{\bar{\Omega}} \hat{v}(x)$, where $\hat{v}(x)$ is the unique positive solution of the boundary-value problem

$$\begin{gathered} d_1\triangle \hat{v} + \hat{v}(f(S(x)) - \hat{v}) = 0, \quad x \in \Omega, \\ \frac{\partial \hat{v}}{\partial \nu} + r_1(x)\hat{v} = 0, \quad x \in \partial\Omega. \end{gathered}$$

Since the proof the above Lemma is quite standard, we omit it here.

Theorem 3.12. *Assume that $f'' \le 0$, then there exist constants K_0 and K_1 depending on f, u^0 and r_i such that if $d_0 \ge K_0$ and $d_1 \ge K_1$, $U_p = (\tilde{u}(x), \tilde{v}(x))$ is asymptotically stable.*

Proof. We prove that $U_p = (\tilde{u}(x), \tilde{v}(x))$ is asymptotically stable by constructing a pair of upper and lower solutions

$$\begin{gathered} \bar{U} = (\bar{u}, \bar{v}) = (p_1(t)\phi_1(x) + \tilde{u}(x), p_2(t)\psi_1(x) + \tilde{v}(x)), \\ \underline{U} = (\underline{u}, \underline{v}) = (\tilde{u}(x) - p_1(t)\phi_1(x), \tilde{v}(x) - p_2(t)\psi_1(x)) \end{gathered}$$

of (3.1), where $p_1(t)$ and $p_2(t)$ are two positive (small) functions and $\phi_1(x) > 0$ and $\psi_1(x) > 0$ are the normalized principal eigenfunctions of the eigenvalue problems

$$\begin{gathered}
-\triangle\phi = \lambda\phi, \quad x \in \Omega, \\
\frac{\partial\phi}{\partial\nu} + r_0(x)\phi = 0, \quad x \in \partial\Omega,
\end{gathered}$$

and

$$\begin{gathered}
-\triangle\psi = \mu\psi, \quad x \in \Omega, \\
\frac{\partial\psi}{\partial\nu} + r_1(x)\psi = 0, \quad x \in \partial\Omega,
\end{gathered}$$

associated with the eigenvalues $\lambda_1 > 0$ and $\mu_1 > 0$. By the definition of upper and lower solutions, $p_i(t)$, $i = 1, 2$, should satisfy

$$\begin{aligned}
&(p_1(t)\phi_1(x) + \tilde{u}(x))_t - d_0\triangle(p_1(t)\phi_1(x) + \tilde{u}(x)) \\
&+ (\tilde{v}(x) - p_2(t)\psi_1(x))f(p_1(t)\phi_1(x) + \tilde{u}(x)) \\
&\geq 0 \\
&\geq (\tilde{u}(x) - p_1(t)\phi_1(x))_t - d_0\triangle(\tilde{u}(x) - p_1(t)\phi_1(x)) \\
&\quad + (p_2(t)\psi_1(x) + \tilde{v}(x))f(\tilde{u}(x) - p_1(t)\phi_1(x)),
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
&(p_2(t)\psi_1(x) + \tilde{v}(x))_t - d_1\triangle(p_2(t)\psi_1(x) + \tilde{v}(x)) \\
&- (p_2(t)\psi_1(x) + \tilde{v}(x))(f(p_1(t)\phi_1(x) + \tilde{u}(x)) - (p_2(t)\psi_1(x) + \tilde{v}(x))) \\
&\geq 0 \\
&\geq (\tilde{v}(x) - p_2(t)\psi_1(x))_t - d_1\triangle(\tilde{v}(x) - p_2(t)\psi_1(x)) \\
&\quad - (\tilde{v}(x) - p_2(t)\psi_1(x))(f(\tilde{u}(x) - p_1(t)\phi_1(x)) - (\tilde{v}(x) - p_2(t)\psi_1(x))).
\end{aligned} \tag{3.15}$$

From the left-hand side of (3.14), we need

$$\begin{aligned}
&p_1'(t)\phi_1(x) - d_0p_1(t)\triangle\phi_1(x) - d_0\triangle\tilde{u}(x) + \tilde{v}(x)f(p_1(t)\phi_1(x) + \tilde{u}(x)) \\
&- p_2(t)\psi_1(x)f(p_1(t)\phi_1(x) + \tilde{u}(x)) \geq 0;
\end{aligned}$$

that is,

$$\begin{aligned}
&p_1'(t)\phi_1(x) + \lambda_1d_0p_1(t)\phi_1(x) - \tilde{v}(x)f(\tilde{u}(x)) + \tilde{v}(x)f(p_1(t)\phi_1(x) + \tilde{u}(x)) \\
&- p_2(t)\psi_1(x)f(p_1(t)\phi_1(x) + \tilde{u}(x)) \geq 0.
\end{aligned}$$

Therefore, we need only

$$p_1'(t)\phi_1(x) + \lambda_1d_0p_1(t)\phi_1(x) - p_2(t)\psi_1(x)f(p_1(t)\phi_1(x) + \tilde{u}(x)) \geq 0.$$

By Taylor's Theorem, we have

$$f(p_1\phi_1 + \tilde{u}) = f(\tilde{u}) + f'(\tilde{u})p_1\phi_1 + \frac{f''(\theta)}{2}p_1^2\phi_1^2 \leq f(\tilde{u}) + f'(\tilde{u})p_1\phi_1.$$

Therefore, we need only

$$p_1'\phi_1 + \lambda_1d_0p_1\phi_1 - p_2\psi_1f(\tilde{u}) - p_2\psi_1f'(\tilde{u})p_1\phi_1 \geq 0,$$

or

$$p_1'\phi_1 + \lambda_1d_0p_1\phi_1 - p_2\psi_1f(\tilde{u}) \geq f'(\tilde{u})p_1p_2\phi_1\psi_1. \tag{3.16}$$

Similarly, from the right-hand side (3.14), we need

$$p_1'\phi_1 + \lambda_1d_0p_1\phi_1 - p_2\psi_1f(\tilde{u}) \geq 0. \tag{3.17}$$

From the left-hand side of (3.15), we need

$$p_2'\psi_1 + \mu_1 d_1 p_2\psi_1 - p_2\psi_1 f(\tilde{u}) - \tilde{v}f'(\tilde{u})p_1\phi_1 \ge f'(\tilde{u})p_1p_2\phi_1\psi_1, \tag{3.18}$$

and from the right-hand side of (3.15), we need

$$p_2'\psi_1 + \mu_1 d_1 p_2\psi_1 - \tilde{v}f(\tilde{u}) + \tilde{v}f(\tilde{u} - p_1\phi_1) - p_2\psi_1 f(\tilde{u}) \ge 0.$$

By Taylor's Theorem, we have

$$f(\tilde{u} - p_1\phi_1) = f(\tilde{u}) - f'(\theta)p_1\phi_1 \ge f(\tilde{u}) - f'(0)p_1\phi_1.$$

Therefore, we need

$$p_2'\psi_1 + \mu_1 d_1 p_2\psi_1 - \tilde{v}f'(0)p_1\phi_1 - p_2\psi_1 f(\tilde{u}) \ge 0. \tag{3.19}$$

Combining (3.16) to (3.19), observing that $\phi_1 \le 1$ and $\psi_1 \le 1$, we need

$$p_1'\phi_1 + \lambda_1 d_0 p_1\phi_1 - p_2\psi_1 f(\tilde{u}) \ge f'(\tilde{u})p_1p_2,$$

and

$$p_2'\psi_1 + \mu_1 d_1 p_2\psi_1 - p_2\psi_1 f(\tilde{u}) - \tilde{v}f'(0)p_1\phi_1 \ge f'(\tilde{u})p_1p_2.$$

Let $\rho = \min_{\bar{\Omega}} \phi_1(x) > 0$, $\sigma = \min_{\bar{\Omega}} \psi_1(x) > 0$, and take $p_1 = p_2 = p$. Then, by Lemma 3.11, we need p to satisfy

$$p' + (\lambda_1 d_0 - \frac{f(\hat{S})}{\rho})p \ge \frac{f'(0)}{\rho}p^2,$$
$$p' + (\mu_1 d_1 - \frac{f(\hat{S})}{\sigma} - \frac{\hat{V}f'(0)}{\sigma})p \ge \frac{f'(0)}{\sigma}p^2.$$

Therefore, if there exists an $\epsilon > 0$ such that

$$\lambda_1 d_0 - \frac{f(\hat{S})}{\rho} \ge \epsilon, \quad \text{and} \quad \mu_1 d_1 - \frac{f(\hat{S})}{\sigma} - \frac{\hat{V}f'(0)}{\sigma} \ge \epsilon,$$

then we need only take p such that

$$p' + \epsilon p \ge Mp^2,$$

where $M = \max\{\frac{f'(0)}{\rho}, \frac{f'(0)}{\sigma}\}$. In particular we take p such that $p' + \epsilon p = Mp^2$, then we have

$$p(t) = \frac{1}{M/\epsilon + (1/p(0) - M/\epsilon)e^{\epsilon t}},$$

where $0 < p(0) < \epsilon/M$.

It is easily seen that $p(t) \to 0$ as $t \to \infty$. Therefore, if the initial values $u_0(x)$ and $v_0(x)$ satisfy

$$\tilde{u}(x) - p(0)\phi_1(x) \le u_0(x) \le \tilde{u}(x) + p(0)\phi_1(x),$$
$$\tilde{v}(x) - p(0)\psi_1(x) \le v_0(x) \le \tilde{v}(x) + p(0)\psi_1(x),$$

then we have

$$|u(x,t) - \tilde{u}(x)| \le p(t)\phi_1(x),$$
$$|v(x,t) - \tilde{v}(x)| \le p(t)\psi_1(x).$$

So we have $u(x,t) \to \tilde{u}(x)$ and $v(x,t) \to \tilde{v}(x)$ as $t \to \infty$. Therefore U_p is asymptotically stable. $\square$

4. Steady States

In this section, we study the existence of steady states of (1.1). The steady states of (1.1) satisfy

$$
\begin{gathered}
d_0\triangle u + F_0(u, v_1, v_2) = 0, \quad x \in \Omega,\\
d_1\triangle v_1 + F_1(u, v_1, v_2) = 0, \quad x \in \Omega,\\
d_2\triangle v_2 + F_2(u, v_1, v_2) = 0, \quad x \in \Omega,\\
\frac{\partial u}{\partial \nu} + r_0(x)u = u^0(x), \quad x \in \partial\Omega,\\
\frac{\partial v_i}{\partial \nu} + r_i(x)v_i = 0,\ i = 1, 2, \quad x \in \partial\Omega.
\end{gathered}
\tag{4.1}
$$

As mentioned before, by Lemma 2.1, (4.1) has a washout solution $U_1 = (u, v_1, v_2) = (S(x), 0, 0)$ with $S(x) > 0$. From Theorem 3.1, we have the following theorem

Theorem 4.1. *Equation* (4.1) *has two nonnegative solutions:*

$$
\begin{gathered}
U_2 = (\tilde{u}(x), \tilde{v}_1(x), 0) \quad \textit{with} \quad \tilde{u}(x) > 0, \quad \tilde{v}_1(x) > 0,\\
U_3 = (\tilde{u}(x), 0, \tilde{v}_2(x)) \quad \textit{with} \quad \tilde{u}(x) > 0, \quad \tilde{v}_2(x) > 0.
\end{gathered}
$$

Theorem 4.2. *Assume that* $f_1 = f_2 = f$, *then* (4.1) *has no positive solution.*

Proof. If $f_1 = f_2 = f$ and (4.1) has a positive solution $U = (u(x), v_1(x), v_2(x))$ with $u(x) > 0$ and $v_i(x) > 0$, then $v_1(x) > 0$ and $v_2(x) > 0$ satisfy

$$
\begin{gathered}
d_1\triangle v_1 + (f(u) - v_1 - v_2)v_1 = 0, \quad x \in \Omega,\\
d_2\triangle v_2 + (f(u) - v_1 - v_2)v_2 = 0, \quad x \in \Omega,\\
\frac{\partial v_i}{\partial \nu} + r_i(x)v_i = 0, \quad i = 1, 2,\ x \in \partial\Omega.
\end{gathered}
$$

Because of the positivity of $v_i(x)$, we can look $v_i(x)$ as the principal eigenfunction of the eigenvalue problem

$$
\begin{gathered}
d_i\triangle \phi + q(x)\phi = \lambda\phi, \quad x \in \Omega,\\
\frac{\partial \phi}{\partial \nu} + r_i(x)\phi = 0, \quad x \in \partial\Omega,
\end{gathered}
$$

with $q(x) = f(u(x)) - \sum_{j=1}^{2} v_j(x)$, associated with the principal eigenvalue $\lambda = \lambda(q(x), d_i, r_i) = 0$. So we have

$$
\lambda(q(x), d_1, r_1) = \lambda(q(x), d_2, r_2).
$$

Since $q(x) = f(u(x)) - \sum_{j=1}^{2} v_j(x)$ is not constant, by Lemma 3.2, this contradicts the assumption $d_1 < d_2$. This completes the proof of the Theorem. $\square$

Same as the proof of Theorem 4.2, we can prove that (4.1) has no solution of the form $U = (0, v_1(x), v_2(x))$ with $v_1(x) > 0$ and $v_2(x) > 0$ if $f_1 = f_2$. Thus we know that if $u^0(x) \not\equiv 0$ and $f_1 = f_2$, (4.1) has only the following three types of solutions

$$
(S(x), 0, 0), \quad (\tilde{u}(x), \tilde{v_1}(x), 0), \quad (\tilde{u}(x), 0, \tilde{v_2}(x)),
$$

with $S(x) > 0$, $\tilde{u}(x) > 0$ and $\tilde{v_i}(x) > 0$.

5. Stability Analysis

In this section, we study the stability of equilibrium solutions of (1.1) under the assumption $f_1 = f_2 = f$.

From Section 4, we know that, if $f_1 = f_2 = f$, all equilibria of (1.1) are $U_1 = (S(x), 0, 0)$, $U_2 = (\tilde{u}(x), \tilde{v}_1(x), 0)$ and $U_3 = (\tilde{u}(x), 0, \tilde{v}_2(x))$, where $S(x) > 0$ is the positive solution of (2.1), $\tilde{u}(x) > 0$ and $\tilde{v}_i(x) > 0$ satisfy

$$\begin{gathered}
d_0 \triangle \tilde{u} - f(\tilde{u})\tilde{v}_i = 0, \quad x \in \Omega, \\
d_i \triangle \tilde{v}_i + \tilde{v}_i(f(\tilde{u}) - \tilde{v}_i) = 0, \quad x \in \Omega, \\
\frac{\partial \tilde{u}}{\partial \nu} + r_0(x)\tilde{u} = u^0(x), \quad x \in \partial\Omega, \\
\frac{\partial \tilde{v}_i}{\partial \nu} + r_i(x)\tilde{v}_i = 0, \quad x \in \partial\Omega.
\end{gathered} \tag{5.1}$$

Observe that, because of the positivity of $\tilde{v}_i(x)$, from the second equation of (5.1), we can look $\tilde{v}_i(x)$ as the principal eigenfunction of the eigenvalue problem

$$\begin{gathered}
d_i \triangle \phi + q(x)\phi = \lambda\phi, \quad x \in \Omega, \\
\frac{\partial \phi}{\partial \nu} + r_i(x)\phi = 0, \quad x \in \partial\Omega,
\end{gathered}$$

with $q(x) = f(\tilde{u}) - \tilde{v}_i$, associated with the principal eigenvalue $\lambda = \lambda(q(x), d_i, r_i) = 0$, $i = 1, 2$.

Now we study the stability of U_i. It is well-known (see [9]) that the stability question for U_i is answered by considering the corresponding eigenvalue problem for the linearized operator around U_i. Namely, let us substitute $U(x,t) = (u(x,t), v_1(x,t), v_2(x,t)) = U_i + W(x,t) = U_i + (w_0(x,t), w_1(x,t), w_2(x,t))$ into (1.1) and then pick up all the terms which are linear in W:

$$\frac{\partial W}{\partial t} = D\triangle W + F'(U_i)W, \tag{5.2}$$

where

$$D = \begin{pmatrix} d_0 & 0 & 0 \\ 0 & d_1 & 0 \\ 0 & 0 & d_2 \end{pmatrix}$$

and

$$F'(U_i) = \begin{pmatrix} -f'(u)(v_1 + v_2) & -f(u) & -f(u) \\ v_1 f'(u) & f(u) - 2v_1 - v_2 & -v_1 \\ v_2 f'(u) & -v_2 & f(u) - v_1 - 2v_2 \end{pmatrix}_{U_i}.$$

Theorem 5.1. *The solution $U_1 = (S(x), 0, 0)$ is unstable.*

The proof is similar to that of Theorem 3.10, therefore, we omit it.

Theorem 5.2. *Assume that $f'' \le 0$, then there exist constants K_0 and K_1 depending on f, u^0 and r_i such that if $d_0 \ge K_0$ and $d_1 \ge K_1$, $U_2 = (\tilde{u}(x), \tilde{v}_1(x), 0)$ is asymptotically stable.*

Proof. From (5.2), the linearized system of (1.1) around U_2 is

$$\begin{gathered}
w_{0t} = d_0 \triangle w_0 - f'(\tilde{u})\tilde{v}_1 w_0 - f(\tilde{u})w_1 - f(\tilde{u})w_2, \quad x \in \Omega, \\
w_{1t} = d_1 \triangle w_1 + \tilde{v}_1 f'(\tilde{u})w_0 + (f(\tilde{u}) - 2\tilde{v}_1)w_1 - \tilde{v}_1 w_2, \quad x \in \Omega, \\
w_{2t} = d_2 \triangle w_2 + (f(\tilde{u}) - \tilde{v}_1)w_2, \quad x \in \Omega, \\
\frac{\partial w_0}{\partial \nu} + r_0(x)w_0 = u^0(x), \quad x \in \partial\Omega, \\
\frac{\partial w_i}{\partial \nu} + r_i(x)w_i = 0,\ i = 1, 2, \quad x \in \partial\Omega.
\end{gathered}$$

Therefore, we need to study the eigenvalue problem:

$$\begin{gathered}
d_0 \triangle w_0 - f'(\tilde{u})\tilde{v}_1 w_0 - f(\tilde{u})w_1 - f(\tilde{u})w_2 = \eta w_0, \quad x \in \Omega, \\
d_1 \triangle w_1 + \tilde{v}_1 f'(\tilde{u})w_0 + (f(\tilde{u}) - 2\tilde{v}_1)w_1 - \tilde{v}_1 w_2 = \eta w_1, \quad x \in \Omega, \\
d_2 \triangle w_2 + (f(\tilde{u}) - \tilde{v}_1)w_2 = \eta w_2, \quad x \in \Omega, \\
\frac{\partial w_0}{\partial \nu} + r_0(x)w_0 = 0, \quad x \in \partial\Omega, \\
\frac{\partial w_i}{\partial \nu} + r_i(x)w_i = 0,\ i = 1, 2, \quad x \in \partial\Omega.
\end{gathered} \tag{5.3}$$

Let η_1 be the largest eigenvalue of (5.3) and (w_0, w_1, w_2) be the corresponding eigenfunctions. If $w_2(x) \not\equiv 0$, then η_1 is an eigenvalue of $d_2\triangle + (f(\tilde{u}) - \tilde{v}_1)$ with Robin boundary condition. So, by Lemma 3.2, we have

$$\eta_1 \le \lambda(f(\tilde{u}) - \tilde{v}_1, d_2, r_2) < \lambda(f(\tilde{u}) - \tilde{v}_1, d_1, r_1) = 0.$$

If $w_2(x) \equiv 0$, it is easily seen that we must have $w_1(x) \not\equiv 0$ and $w_0(x) \not\equiv 0$. In this case, η_1 satisfies

$$\begin{gathered}
d_0 \triangle w_0 - f'(\tilde{u})\tilde{v}_1 w_0 - f(\tilde{u})w_1 = \eta_1 w_0, \quad x \in \Omega, \\
d_1 \triangle w_1 + \tilde{v}_1 f'(\tilde{u})w_0 + (f(\tilde{u}) - 2\tilde{v}_1)w_1 = \eta_1 w_1, \quad x \in \Omega, \\
\frac{\partial w_0}{\partial \nu} + r_0(x)w_0 = 0, \quad x \in \partial\Omega, \\
\frac{\partial w_1}{\partial \nu} + r_1(x)w_1 = 0, \quad x \in \partial\Omega.
\end{gathered}$$

Therefore, η_1 is also an eigenvalue of the eigenvalue problem

$$\begin{gathered}
d_0 \triangle w_0 - f'(\tilde{u})\tilde{v} w_0 - f(\tilde{u})w_1 = \eta w_0, \quad x \in \Omega, \\
d_1 \triangle w_1 + \tilde{v} f'(\tilde{u})w_0 + (f(\tilde{u}) - 2\tilde{v})w_1 = \eta w_1, \quad x \in \Omega, \\
\frac{\partial w_0}{\partial \nu} + r_0(x)w_0 = 0, \quad x \in \partial\Omega, \\
\frac{\partial w_1}{\partial \nu} + r_1(x)w_1 = 0, \quad x \in \partial\Omega.
\end{gathered} \tag{5.4}$$

This is the linearized system of (3.1) around U_p. From Theorem 3.12, we know that all eigenvalues of (5.4) are negative. Therefore, $\eta_1 < 0$. Hence all eigenvalues of (5.1) are negative. Therefore, U_2 is asymptotically stable. $\square$

Theorem 5.3. *The solution $U_3 = (\tilde{u}(x), 0, \tilde{v}_2(x))$ is unstable.*

Proof. As in the proof of the previous theorem, we need to study the eigenvalue problem

$$\begin{gathered} d_0\triangle w_0 - f'(\tilde u)\tilde v_2 w_0 - f(\tilde u)w_1 - f(\tilde u)w_2 = \eta w_0, \quad x\in\Omega,\\ d_1\triangle w_1 + (f(\tilde u)-\tilde v_2)w_1 = \eta w_1, \quad x\in\Omega,\\ d_2\triangle w_2 + \tilde v_2 f'(\tilde u)w_0 - \tilde v_2 w_1 + (f(\tilde u)-2\tilde v_2)w_2 = \eta w_2, \quad x\in\Omega,\\ \frac{\partial w_0}{\partial\nu} + r_0(x)w_0 = 0, \quad x\in\partial\Omega,\\ \frac{\partial w_i}{\partial\nu} + r_i(x)w_i = 0,\ i=1,2, \quad x\in\partial\Omega. \end{gathered} \tag{5.5}$$

Let $\lambda_1 = \lambda(f(\tilde u)-\tilde v_2, d_1, r_1)$ and $\phi_1(x)$ be the principal eigenpair of

$$\begin{gathered} d_1\triangle\phi + (f(\tilde u)-\tilde v_2)\phi = \lambda\phi, \quad x\in\Omega,\\ \frac{\partial\phi}{\partial\nu} + r_1(x)\phi = 0, \quad x\in\partial\Omega, \end{gathered}$$

then we have $\lambda_1 > \lambda(f(\tilde u)-\tilde v_2, d_2, r_2) = 0$. Let $(\tilde w_0, \tilde w_2)$ be the solution of the following linear boundary value problem

$$\begin{gathered} d_0\triangle w_0 - (f'(\tilde u)\tilde v_2 + \lambda_1)w_0 - f(\tilde u)w_2 = f(\tilde u)\phi, \quad x\in\Omega,\\ d_2\triangle w_2 + \tilde v_2 f'(\tilde u)w_0 + (f(\tilde u)-2\tilde v_2-\lambda_1)w_2 = \tilde v_2\phi, \quad x\in\Omega,\\ \frac{\partial w_0}{\partial\nu} + r_0(x)w_0 = 0, \quad x\in\partial\Omega,\\ \frac{\partial w_2}{\partial\nu} + r_2(x)v_2 = 0, \quad x\in\partial\Omega, \end{gathered}$$

then it is easily seen that $\lambda_1 > 0$ is an eigenvalue of (5.5) with eigenfunction $(w_0, w_1, w_2) = (\tilde w_0, \phi_1, \tilde w_2)$. Therefore, U_3 is unstable. □

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