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ASYMPTOTIC BEHAVIOR OF A PREDATOR-PREY DIFFUSION SYSTEM WITH TIME DELAYS

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Abstract. In this paper, we study a class of reaction-diffusion systems with time delays, which models the dynamics of predator-prey species. The global asymptotic convergence is established by the upper-lower solutions and iteration method in terms of the rate constants of the reaction function, independent of the time delays and the effect of diffusion

1. Introduction

The purpose of this paper is to study the asymptotic behavior of solutions to the predator-prey diffusion system with time delays:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \Delta u + u[a_1 - b_1 u - \int_0^\infty f_1(\tau) u(t-\tau, x) d\tau - d_1 v(t - r_2, x)],\\ &\qquad t > 0,\ x \in \Omega,\\ \frac{\partial v}{\partial t} &= \Delta v + v[a_2 - b_2 v - \int_0^\infty f_2(\tau) v(t-\tau, x) d\tau + d_2 u(t - r_1, x)],\end{aligned} \tag{1.1}$$

subjected to the boundary conditions

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad t > 0,\ x \in \partial\Omega, \tag{1.2}$$

and to the nonnegative initial conditions

$$u(t,x) = \phi_1(t,x),\ v(t,x) = \phi_2(t,x), \quad i = 1,2,\ t \le 0,\ x \in \overline{\Omega}, \tag{1.3}$$

where $\Omega \subseteq \mathbb{R}^N$ $(N \ge 1)$ is a bounded domain with smooth boundary $\partial\Omega$, and $\partial/\partial n$ denotes differentiation in the direction of the outward normal. a_i, b_i, d_i and $r_i (i = 1, 2)$ are positive constants. $f_i \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, and the integral part means the hereditary term concerning the effect of the past history on the present growth rate. $\phi_i \in C^1((-\infty, 0] \times \overline{\Omega})$ is bounded nonnegative.

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We write $f_i = f_i^+ - f_i^- (i = 1, 2)$, where $f_i^+(s) = \max(0, f(s))$, and $f_i^-(s) = \max(0, -f_i(s))$ for $s \geq 0$. We set

$$c_i^+ = \int_0^\infty f_i^+(s)ds, \quad c_i^- = \int_0^\infty f_i^-(s)ds \quad i = 1, 2.$$

Throughout the paper, we assume that

$$b_1 > \int_0^\infty |f_1(s)|ds, \quad b_2 > \int_0^\infty |f_2(s)|ds, \tag{1.4}$$

and

$$\begin{aligned} &\frac{d_2 c_2^+}{(b_1 - c_1^+ - c_1^-)(b_2 - c_2^+ - c_2^-) - d_1 d_2} \\ &< \frac{a_2}{a_1} < \frac{(b_1 - c_1^+ - c_1^-)(b_2 - c_2^+ - c_2^-) - d_1 d_2}{d_1(b_1 - c_1^-)}, \end{aligned} \tag{1.5}$$

Our result can be stated as follows.

Theorem 1.1. *Assume that f_1 and f_2 belong to $C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ and (1.4)–(1.5) hold. Then for every $\phi_i \in C^1((\infty, 0]) \times \overline{\Omega}$ with $\phi_i(0, x) \not\equiv 0$, the solution of (1.1)–(1.3) satisfies*

$$\lim_{t\to\infty} u(t, x) = \frac{a_1(b_2 + c_2^+ - c_2^-) - a_2 d_1}{(b_1 + c_1^+ - c_1^-)(b_2 + c_2^+ - c_2^-) + d_1 d_2}, \tag{1.6}$$

uniformly for $x \in \overline{\Omega}$. Also

$$\lim_{t\to\infty} v(t, x) = \frac{a_2(b_1 + c_1^+ - c_1^-) + a_1 d_2}{(b_1 + c_1^+ - c_1^-)(b_2 + c_2^+ - c_2^-) + d_1 d_2}, \tag{1.7}$$

uniformly for $x \in \overline{\Omega}$.

Remark. If $f_1 \equiv 0$ and $f_1 \equiv 0$, then

$$\lim_{t\to\infty} u(t, x) = \frac{a_1 b_2 + a_2 d_1}{b_1 b_2 + d_1 d_2}, \quad \text{and} \quad \lim_{t\to\infty} v(t, x) = \frac{a_2 b_1 + a_1 d_2}{b_1 b_2 + d_1 d_2},$$

uniformly for $x \in \overline{\Omega}$, which coincides with the result of [4].

Let us introduce the following result (see [7]) on the asymptotic behavior of the diffusion logistic equation with time delays, which plays an important role in the proof of Theorem.

$$\begin{gathered} \frac{\partial u}{\partial t} = \Delta u + u[a - bu - \int_0^\infty f(\tau)u(t-\tau, x)d\tau], \quad t > 0,\ x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, \quad t > 0,\ x \in \partial\Omega, \\ u(t, x) = \phi(t, x), \quad t \leq 0,\ x \in \overline{\Omega}, \end{gathered} \tag{1.8}$$

where a and b are positive constants, $\phi \in C^1((-\infty, 0] \times \overline{\Omega})$ is a bounded nonnegative function.

Lemma 1.2. *Assume that $f \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ and $b > \int_0^\infty |f(s)|ds$. Then (1.8) has a unique bounded nonnegative solution. Moreover, if $\phi(0, x) \not\equiv 0$, then $u(t, x) > 0$ for all $(t, x) \in (0, \infty) \times \overline{\Omega}$ and*

$$\lim_{t\to\infty} u(t, x) = \frac{a}{b + \int_0^\infty f(s)ds},$$

uniformly for $x \in \overline{\Omega}$.

Reaction-diffusion systems with delays have been studied by many authors. However, most of the systems are mixed quasimonotone, and most of the discussions are in the framework of semi-group theory of dynamical systems [2, 3, 8, 9]. The method of upper and lower solutions and its associated monotone iterations have been used to investigate the dynamic property of the system, which is mixed quasimonotone with discrete delays [1, 5, 6]. In this paper, the method of proof is via successive improvement of upper-lower solutions of some suitable systems, and the fact that we are dealing with system (1.1) without mixed quasimonotone forces us to develop some significance in the process of proof.

2. Proof of Main Results

In this section, we first introduce the following existence-comparison result for the predator-prey system (1.1)–(1.3).

Definition [5] A pair of smooth functions $(\tilde{u}, \tilde{v})$ and $(\hat{u}, \hat{v})$ are called upper-lower solutions of (1.1)–(1.3), if $\tilde{u} \geq \hat{u}$, $\tilde{v} \geq \hat{v}$ in $\mathbb{R}^1 \times \overline{\Omega}$, and if for all $\hat{u} \leq \psi_1 \leq \tilde{u}$, $\hat{v} \leq \psi_2 \leq \tilde{v}$, the following differential inequalities hold.

$$
\begin{gathered}
\frac{\partial \tilde{u}}{\partial t} - \Delta \tilde{u} \geq \tilde{u}[a_1 - b_1 \tilde{u} - \int_0^\infty f_1(\tau)\psi_1(t-\tau, x)d\tau - d_1 \hat{v}(t - r_2, x)], \\
t > 0,\ x \in \Omega, \\
\frac{\partial \tilde{v}}{\partial t} - \Delta \tilde{v} \geq \tilde{v}[a_2 - b_2 \tilde{v} - \int_0^\infty f_2(\tau)\psi_2(t-\tau, x)d\tau + d_2 \tilde{u}(t - r_1, x)], \\
\frac{\partial \hat{u}}{\partial t} - \Delta \hat{u} \leq \hat{u}[a_1 - b_1 \hat{u} - \int_0^\infty f_1(\tau)\psi_1(t-\tau, x)d\tau - d_1 \tilde{v}(t - r_2, x)], \\
\frac{\partial \hat{v}}{\partial t} - \Delta \hat{v} \leq \hat{v}[a_1 - b_1 \hat{v} - \int_0^\infty f_2(\tau)\psi_2(t-\tau, x)d\tau + d_2 \hat{u}(t - r_1, x)], \\
\frac{\partial \hat{u}}{\partial n} \leq 0 \leq \frac{\partial \tilde{u}}{\partial n},\ \frac{\partial \hat{v}}{\partial n} \leq 0 \leq \frac{\partial \tilde{v}}{\partial n}, \quad t > 0, x \in \partial\Omega, \\
\hat{u}(t,x) \leq \phi_1(t,x) \leq \tilde{u}(t,x), \quad \hat{v}(t,x) \leq \phi_2(t,x) \leq \tilde{v}(t,x), \quad t \leq 0,\ x \in \overline{\Omega}.
\end{gathered} \tag{2.1}
$$

With these definitions of upper-lower solutions, we can state the following lemma.

Lemma 2.1 ([5]). *If there exists a pair of upper-lower solutions* $(\tilde{u}, \tilde{v})$, $(\hat{u}, \hat{v})$ *of* (1.1)–(1.3). *Then the problem* (1.1)–(1.3) *has a unique solution* (u^*, v^*) *satisfying* $\hat{u} \leq u^* \leq \tilde{u}$, $\hat{v} \leq v^* \leq \tilde{v}$.

For a given $\phi = (\phi_1, \phi_2)$, let M_1, M_2 be constants such that

$$
M_1 \geq \max\Big\{\|\phi_1\|, \frac{a_1}{b_1 - \int_0^\infty |f_1(s)|ds}\Big\}, \quad M_2 \geq \max\Big\{\|\phi_2\|, \frac{a_2 + d_2 M_1}{b_2 - \int_0^\infty |f_2(s)|ds}\Big\}
$$

where $\|\phi_i\| = \sup_{(t,x)\in(-\infty,0]\times\overline{\Omega}} |\phi_i(t,x)|$, $i = 1, 2$. Then $(0,0)$ and (M_1, M_2) are clearly a pair of lower-upper solutions of (1.1)–(1.3). By Lemma 2.1, a unique global nonnegative solution (u, v) to (1.1)–(1.3) exists and satisfies $0 \leq u \leq M_1, 0 \leq v \leq M_2$, moreover (u, v) is positive in $(0, +\infty) \times \overline{\Omega}$ if $\phi_i(0, x) \not\equiv 0 (i = 1, 2)$ by maximal principle.

Define $\overline{u}_1(t,x)$ by

$$\begin{gathered}\frac{\partial \overline{u}_1}{\partial t} = \Delta \overline{u}_1 + \overline{u}_1[a_1 - b_1\overline{u}_1 + \int_0^\infty f_1^-(\tau)\overline{u}_1(t-\tau,x)d\tau], \quad t>0,\ x\in\Omega,\\ \frac{\partial \overline{u}_1}{\partial n} = 0, \quad t>0,\ x\in\partial\Omega,\\ \overline{u}_1(t,x) = M_1, \quad t\le 0,\ x\in\overline{\Omega}.\end{gathered} \tag{2.2}$$

By Lemma 1.2, we have

$$\lim_{t\to\infty}\overline{u}_1(t,x) = \frac{a_1}{b_1 - c_1^-} = \overline{\alpha}_1, \quad \text{uniformly for } x\in\overline{\Omega}.$$

So, for all sufficiently small $\varepsilon>0$, there exists a $t_1>0$, such that

$$\max_{x\in\overline{\Omega}}\overline{u}_1(t,x) < \overline{\alpha}_1 + \varepsilon, \quad \text{for } t>t_1. \tag{2.3}$$

Define $\overline{v}_1(t,x)$ by

$$\begin{gathered}\frac{\partial \overline{v}_1}{\partial t} = \Delta \overline{v}_1 + \overline{v}_1[a_2 - b_2\overline{v}_1 + \int_0^\infty f_2^-(\tau)\overline{v}_1(t-\tau,x)d\tau + d_2\overline{u}_1], \quad t>t_1,\ x\in\Omega,\\ \frac{\partial \overline{v}_1}{\partial n} = 0, \quad t>t_1,\ x\in\partial\Omega,\\ \overline{v}_1(t,x) = M_2, \quad t\le t_1,\ x\in\overline{\Omega}.\end{gathered} \tag{2.4}$$

It is easy to check that $(0,0)$ and $(\overline{u}_1,\overline{v}_1)$ are the lower and upper solutions of (1.1)–(1.3). Therefore, Lemma 2.1 implies

$$0\le u\le \overline{u}_1, \quad 0\le v\le \overline{v}_1.$$

From (2.3) and (2.4), it follows that

$$\frac{\partial \overline{v}_1}{\partial t} \le \Delta \overline{v}_1 + \overline{v}_1[a_2 - b_2\overline{v}_1 + \int_0^\infty f_2^-(\tau)\overline{v}_1(t-\tau,x)d\tau + d_2(\overline{\alpha}_1+\varepsilon)].$$

By the comparison principle,

$$\overline{v}_1 \le \overline{V}_1,$$

where $\overline{V}_1$ is the solution of the problem

$$\begin{gathered}\frac{\partial \overline{V}_1}{\partial t} = \Delta \overline{V}_1 + \overline{V}_1[a_2 - b_2\overline{V}_1 + \int_0^\infty f_2^-(\tau)\overline{V}_1(t-\tau,x)d\tau + d_2(\overline{\alpha}_1+\varepsilon)],\\ t>t_1,\ x\in\Omega,\\ \frac{\partial \overline{V}_1}{\partial n} = 0, \quad t>t_1,\ x\in\partial\Omega,\\ \overline{V}_1(t,x) = M_2, \quad t\le t_1,\ x\in\overline{\Omega}.\end{gathered}$$

From Lemma 1.2,

$$\lim_{t\to\infty}\overline{V}_1(t,x) = \frac{a_2 + d_2\overline{\alpha}_1}{b_2 - c_2^-} + \varepsilon\frac{d_2}{b_2 - c_2^-}, \quad \text{uniformly for } x\in\overline{\Omega}.$$

So, for all sufficiently small ε, there exists a $t_2>t_1$ such that

$$\max_{x\in\overline{\Omega}}\overline{v}_1(t,x) < \overline{\beta}_1 + \varepsilon, \quad \text{for } t>t_2, \tag{2.5}$$

where $\overline{\beta}_1 = (a_2 + d_2\overline{\alpha}_1)/(b_2 - c_2^-)$. Define $\underline{u}_1$ by

$$\begin{gathered}
\frac{\partial \underline{u}_1}{\partial t} = \Delta \underline{u}_1 + \underline{u}_1\Big[a_1 - b_1\underline{u}_1 + \int_0^\infty f_1^-(\tau)\underline{u}_1(t-\tau,x)d\tau \\
- \int_0^\infty f_1^+(\tau)\overline{u}_1(t-\tau,x)d\tau - d_1\overline{v}_1(t-r_2,x)\Big], \quad t > t_2,\ x \in \Omega, \\
\frac{\partial \underline{u}_1}{\partial n} = 0, \quad t > t_2,\ x \in \partial\Omega, \\
\underline{u}_1(t,x) = \frac{1}{2}u(t,x), \quad (t,x) \in (-\infty, t_2] \times \overline{\Omega}.
\end{gathered} \tag{2.6}$$

From (2.5) and (2.6), for $t > t_2$, $x \in \Omega$ we have

$$\frac{\partial \underline{u}_1}{\partial t} \geq \Delta \underline{u}_1 + \underline{u}_1[a_1 - b_1\underline{u}_1 + \int_0^\infty f_1^-(\tau)\underline{u}_1(t-\tau,x)d\tau - c_1^+(\overline{\alpha}_1 + \varepsilon) - d_1(\overline{\beta}_1 + \varepsilon)].$$

By the comparison principle,

$$\underline{u}_1 \geq \underline{U}_1, \quad t > t_2,\ x \in \Omega,$$

where $\underline{U}_1$ is defined by

$$\begin{gathered}
\frac{\partial \underline{U}_1}{\partial t} = \Delta \underline{U}_1 + \underline{U}_1[a_1 - b_1\underline{U}_1 + \int_0^\infty f_1^-(\tau)\underline{U}_1(t-\tau,x)d\tau \\
- c_1^+(\overline{\alpha}_1 + \varepsilon) - d_1(\overline{\beta}_1 + \varepsilon)], \quad t > t_2,\ x \in \Omega, \\
\frac{\partial \underline{U}_1}{\partial n} = 0, \quad t > t_2,\ x \in \partial\Omega, \\
\underline{U}_1(t,x) = \frac{1}{2}u(t,x), \quad (t,x) \in (-\infty, t_2] \times \overline{\Omega}.
\end{gathered}$$

By (1.5) with ε sufficiently small,

$$a_1 - c_1^+(\overline{\alpha}_1 + \varepsilon) - d_1(\overline{\beta}_1 + \varepsilon) > 0.$$

Thus from Lemma 1.2, we have

$$\lim_{t\to\infty} \underline{U}_1(t,x) = \frac{a_1 - c_1^+\overline{\alpha}_1 - d_1\overline{\beta}_1}{b_1 - c_1^-} - \varepsilon\frac{c_1^+ + d_1}{b_1 - c_1^-}, \quad \text{uniformly for } x \in \overline{\Omega}.$$

Hence for any sufficiently small $\varepsilon > 0$, there exists a $t_3 > t_2$ such that

$$\min_{x\in\overline{\Omega}} \underline{u}_1(t,x) > \underline{\alpha}_1 - \varepsilon, \quad t > t_3, \tag{2.7}$$

where $\underline{\alpha}_1 = (a_1 - c_1^+\overline{\alpha}_1 - d_1\overline{\beta}_1)/(b_1 - c_1^-)$. Define $\underline{v}_1$ by

$$\begin{gathered}
\frac{\partial \underline{v}_1}{\partial t} = \Delta \underline{v}_1 + \underline{v}_1[a_2 - b_2\underline{v}_1 + \int_0^\infty f_2^-(\tau)\underline{v}_1(t-\tau,x)d\tau \\
- \int_0^\infty f_2^+(\tau)\overline{v}_1(t-\tau,x)d\tau + d_2\underline{u}_1(t-r_1,x)], \quad t > t_3, x \in \Omega, \\
\frac{\partial \underline{v}_1}{\partial n} = 0, \quad t > t_3,\ x \in \partial\Omega, \\
\underline{v}_1(t,x) = \frac{1}{2}v(t,x), \quad (t,x) \in (-\infty, t_3] \times \overline{\Omega}.
\end{gathered} \tag{2.8}$$

It is easy to check that $(\overline{u}_1, \overline{v}_1)$ and $(\underline{u}_1, \underline{v}_1)$ are the upper and lower solutions of (1.1)–(1.3), and from Lemma 2.1 we get

$$\underline{u}_1 \leq u \leq \overline{u}_1, \quad \underline{v}_1 \leq v \leq \overline{v}_1.$$

From (2.5), (2.7) and (2.8), we have

$$\frac{\partial \underline{v}_1}{\partial t} \geq \Delta \underline{v}_1 + \underline{v}_1[a_2 - b_2\underline{v}_1 + \int_0^\infty f_2^-(\tau)\underline{v}_1(t-\tau,x)d\tau - c_2^+(\overline{\beta}_1+\varepsilon) + d_2(\underline{\alpha}_1 - \varepsilon)].$$

By the comparison principle,

$$\underline{v}_1 \geq \underline{V}_1, \quad t > t_3,\ x \in \Omega,$$

where $\underline{V}_1$ is defined by

$$\begin{gathered}\frac{\partial \underline{V}_1}{\partial t} = \Delta \underline{V}_1 + \underline{V}_1[a_2 - b_2\underline{V}_1 + \int_0^\infty f_2^-(\tau)\underline{V}_1(t-\tau,x)d\tau \\ - c_2^+(\overline{\beta}_1+\varepsilon) + d_2(\underline{\alpha}_1-\varepsilon)], \quad t > t_3,\ x \in \Omega, \\ \frac{\partial \underline{V}_1}{\partial n} = 0, \quad t > t_3,\ x \in \partial\Omega, \\ \underline{V}_1(t,x) = \frac{1}{2}v(t,x), \quad (t,x) \in (-\infty, t_3] \times \overline{\Omega}.\end{gathered}$$

Note that from (1.5),

$$a_2 - c_2^+(\overline{\beta}_1+\varepsilon) + d_2(\underline{\alpha}_1-\varepsilon) > 0.$$

for sufficiently small ε. From Lemma 1.2, we get

$$\lim_{t\to\infty} \underline{V}_1(t,x) = \frac{a_2 - c_2^+\overline{\beta}_1 + d_2\underline{\alpha}_1}{b_2 - c_2^-} - \varepsilon\frac{c_2^+ + d_2}{b_2 - c_2^-},$$

uniformly for $x \in \overline{\Omega}$. So for any sufficiently small ε, there exists a $t_4 > t_3$ such that

$$\min_{x\in\overline{\Omega}} \underline{v}_1(t,x) > \underline{\beta}_1 - \varepsilon, \quad t > t_4, \tag{2.9}$$

where $\underline{\beta}_1 = \frac{a_2 - c_2^+\overline{\beta}_1 + d_2\underline{\alpha}_1}{b_2 - c_2^-}$. Hence for all sufficiently small ε, we can conclude

$$0 < \underline{\alpha}_1 \leq \liminf_{t\to\infty} \min_{x\in\overline{\Omega}} u(t,x) \leq \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} u(t,x) \leq \overline{\alpha}_1, \tag{2.10}$$

and

$$0 < \underline{\beta}_1 \leq \liminf_{t\to\infty} \min_{x\in\overline{\Omega}} v(t,x) \leq \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} v(t,x) \leq \overline{\beta}_1. \tag{2.11}$$

Define $\overline{u}_2$ by

$$\begin{gathered}\frac{\partial \overline{u}_2}{\partial t} = \Delta \overline{u}_2 + \overline{u}_2[a_1 - b_1\overline{u}_2 + \int_0^\infty f_1^-(\tau)\overline{u}_2(t-\tau,x)d\tau \\ - \int_0^\infty f_1^+(\tau)\underline{u}_1(t-\tau,x)d\tau - d_1\underline{v}_1(t-r_2,x)], \qquad t > t_4, x \in \Omega, \\ \frac{\partial \overline{u}_2}{\partial n} = 0, \quad t > t_4,\ x \in \partial\Omega, \\ \overline{u}_2(t,x) = M_1, \quad (t,x) \in (-\infty, t_4] \times \overline{\Omega}.\end{gathered} \tag{2.12}$$

From (2.7), (2.9) and (2.12), for $t > t_4$, we have

$$\frac{\partial \overline{u}_2}{\partial t} \leq \Delta \overline{u}_2 + \overline{u}_2[a_1 - b_1\overline{u}_2 + \int_0^\infty f_1^-(\tau)\overline{u}_2(t-\tau,x)d\tau - c_1^+(\underline{\alpha}_1 - \varepsilon) - d_1(\underline{\beta}_1 - \varepsilon)].$$

By the comparison principle, we get $\overline{u}_2 \le \overline{U}_1$, $t > t_4$, where $\overline{U}_1$ is defined by

$$\begin{gathered}\frac{\partial \overline{U}_1}{\partial t} \le \Delta \overline{U}_1 + \overline{U}_1[a_1 - b_1\overline{U}_1 + \int_0^\infty f_1^-(\tau)\overline{U}_1(t-\tau,x)d\tau \\ - c_1^+(\underline{\alpha}_1 - \varepsilon) - d_1(\underline{\beta}_1 - \varepsilon)], \quad t > t_4,\ x \in \Omega, \\ \frac{\partial \overline{U}_1}{\partial n} = 0, \quad t > t_4,\ x \in \partial\Omega, \\ \overline{U}_1(t,x) = K_1, \quad (t,x) \in (-\infty, t_4] \times \overline{\Omega}.\end{gathered}$$

For sufficiently small ε, It is easy to show that

$$a_1 - c_1^+(\underline{\alpha}_1 - \varepsilon) - d_1(\underline{\beta}_1 - \varepsilon) > 0.$$

Thus, from lemma 1.2, we have

$$\lim_{t\to\infty} \overline{U}_1(t,x) = \frac{a_1 - c_1^+\underline{\alpha}_1 - d_1\underline{\beta}_1}{b_1 - c_1^-} + \varepsilon\frac{c_1^+ + d_1}{b_1 - c_1^-}, \quad \text{uniformly for } x \in \overline{\Omega}.$$

Hence, for any sufficiently small $\varepsilon > 0$, there exists a $t_5 > t_4$ such that

$$\max_{x\in\overline{\Omega}} \overline{u}_2(t,x) < \overline{\alpha}_2 + \varepsilon, \quad t > t_5, \tag{2.13}$$

where $\overline{\alpha}_2 = \frac{a_1 - c_1^+\underline{\alpha}_1 - d_1\underline{\beta}_1}{b_1 - c_1^-}$.

Define $\overline{v}_2$ by

$$\begin{gathered}\frac{\partial \overline{v}_2}{\partial t} = \Delta \overline{v}_2 + \overline{v}_2[a_2 - b_2\overline{v}_2 + \int_0^\infty f_2^-(\tau)\overline{v}_2(t-\tau,x)d\tau \\ - \int_0^\infty f_2^+(\tau)\underline{v}_1(t-\tau,x)d\tau + d_2\overline{u}_2(t-r_1,x)], \quad t > t_5, x \in \Omega, \\ \frac{\partial \overline{v}_2}{\partial n} = 0, \quad t > t_5,\ x \in \partial\Omega, \\ \overline{v}_2(t,x) = M_2, \quad (t,x) \in (-\infty, t_5] \times \overline{\Omega}.\end{gathered} \tag{2.14}$$

It is easy to check that $(\underline{u}_1, \underline{v}_1)$ and $(\overline{u}_2, \overline{v}_2)$ are the lower and upper solutions of (1.1)–(1.3), and thus from Lemma 2.1, we get

$$\underline{u}_1 \le u \le \overline{u}_2, \quad \underline{v}_1 \le v \le \overline{v}_2$$

From (2.9), (2.13) and (2.14), for $t > t_5$, we have

$$\frac{\partial \overline{v}_2}{\partial t} \le \Delta \overline{v}_2 + \overline{v}_2[a_2 - b_2\overline{v}_2 + \int_0^\infty f_2^-(\tau)\overline{v}_2(t-\tau,x)d\tau - c_2^+(\underline{\beta}_1 - \varepsilon) + d_2(\underline{\alpha}_2 + \varepsilon)].$$

By the comparison principle, we get $\overline{v}_2 \le \overline{V}_2$, $t > t_5$, where $\overline{V}_2$ is defined by

$$\begin{gathered}\frac{\partial \overline{V}_2}{\partial t} = \Delta \overline{V}_2 + \overline{V}_2[a_2 - b_2\overline{V}_2 + \int_0^\infty f_2^-(\tau)\overline{V}_2(t-\tau,x)d\tau \\ - c_2^+(\underline{\beta}_1 - \varepsilon) + d_2(\overline{\alpha}_2 + \varepsilon)], \quad t > t_5,\ x \in \Omega, \\ \frac{\partial \overline{V}_2}{\partial n} = 0, \quad t > t_5,\ x \in \partial\Omega, \\ \overline{V}_2(t,x) = M_2, \quad (t,x) \in (-\infty, t_5] \times \overline{\Omega}.\end{gathered}$$

For sufficiently small ε, it is easy to show that

$$a_2 - c_2^+(\underline{\beta}_1 - \varepsilon) - d_2(\overline{\alpha}_2 + \varepsilon) > 0.$$

Thus from lemma 1.2, we have

$$\lim_{t\to\infty} \overline{V}_2(t,x) = \frac{a_2 - c_2^+\underline{\beta}_1 + d_2\overline{\alpha}_2}{b_2 - c_2^-} + \varepsilon\frac{c_2^+ + d_2}{b_2 - c_2^-}, \quad \text{uniformly for } x \in \overline{\Omega}.$$

Hence for any sufficiently small $\varepsilon > 0$, there exists $t_6 > t_5$ such that

$$\max_{x\in\overline{\Omega}} \overline{v}_2(t,x) < \overline{\beta}_2 + \varepsilon, \quad t > t_6, \tag{2.15}$$

where $\overline{\beta}_2 = \frac{a_2 - c_2^+\underline{\beta}_1 + d_2\overline{\alpha}_2}{b_2 - c_2^-}$. Define $\underline{u}_2$ by

$$\begin{gathered}
\frac{\partial \underline{u}_2}{\partial t} = \Delta\underline{u}_2 + \underline{u}_2[a_1 - b_1\underline{u}_2 + \int_0^\infty f_1^-(\tau)\underline{u}_2(t-\tau,x)d\tau \\
- \int_0^\infty f_1^+(\tau)\overline{u}_2(t-\tau,x)d\tau - d_1\overline{v}_2(t-r_2,x)], \quad t > t_6, x \in \Omega, \\
\frac{\partial \underline{u}_2}{\partial n} = 0, \quad t > t_6,\ x \in \partial\Omega, \\
\underline{u}_2(t,x) = \frac{1}{2}u(t,x), \quad (t,x) \in (-\infty, t_6] \times \overline{\Omega}.
\end{gathered} \tag{2.16}$$

From (2.13), (2.15) and (2.16), for $t > t_6, x \in \Omega$ we get

$$\frac{\partial \underline{u}_2}{\partial t} \ge \Delta\underline{u}_2 + \underline{u}_2[a_1 - b_1\underline{u}_2 + \int_0^\infty f_1^-(\tau)\underline{u}_2(t-\tau,x)d\tau - c_1^+(\overline{\alpha}_2+\varepsilon) - d_1(\overline{\beta}_2+\varepsilon)].$$

By the comparison principle,

$$\underline{u}_2 \ge \underline{U}_2, \quad t > t_6,\ x \in \Omega,$$

where $\underline{U}_2$ is defined by

$$\begin{gathered}
\frac{\partial \underline{U}_2}{\partial t} = \Delta\underline{U}_2 + \underline{U}_2[a_1 - b_1\underline{U}_2 + \int_0^\infty f_1^-(\tau)\underline{U}_2(t-\tau,x)d\tau \\
- c_1^+(\overline{\alpha}_2+\varepsilon) - d_1(\overline{\beta}_2+\varepsilon)], \quad t > t_6,\ x \in \Omega, \\
\frac{\partial \underline{U}_2}{\partial n} = 0, \quad t > t_6,\ x \in \partial\Omega, \\
\underline{U}_2(t,x) = \frac{1}{2}u(t,x), \quad (t,x) \in (-\infty, t_6] \times \overline{\Omega}.
\end{gathered}$$

For sufficiently small ε, we can get

$$a_1 - c_1^+(\overline{\alpha}_2+\varepsilon) - d_1(\overline{\beta}_2+\varepsilon) > 0.$$

Thus from Lemma 1.2, we have

$$\lim_{t\to\infty} \underline{U}_2(t,x) = \frac{a_1 - c_1^+\overline{\alpha}_2 - d_1\overline{\beta}_2}{b_1 - c_1^-} - \varepsilon\frac{c_1^+ + d_1}{b_1 - c_1^-}, \text{ uniformly for } x \in \overline{\Omega}.$$

Hence, for any sufficiently small $\varepsilon > 0$, there exists a $t_7 > t_6$ such that

$$\min_{x\in\overline{\Omega}} \underline{u}_2(t,x) > \underline{\alpha}_2 - \varepsilon, \quad t > t_3, \tag{2.17}$$

where $\underline{\alpha}_2 = \frac{a_1 - c_1^+ \overline{\alpha}_2 - d_1 \overline{\beta}_2}{b_1 - c_1^-}$. Define $\underline{v}_2$ by

$$\begin{gathered} \frac{\partial \underline{v}_2}{\partial t} = \Delta \underline{v}_2 + \underline{v}_2 [a_2 - b_2 \underline{v}_2 + \int_0^\infty f_2^-(\tau) \underline{v}_2(t-\tau, x) d\tau \\ - \int_0^\infty f_2^+(\tau) \overline{v}_2(t-\tau, x) d\tau + d_2 \underline{u}_2(t - r_1, x)], \quad t > t_7, x \in \Omega, \\ \frac{\partial \underline{v}_2}{\partial n} = 0, \quad t > t_7,\ x \in \partial\Omega, \\ \underline{v}_2(t,x) = \frac{1}{2} v(t,x), \quad (t,x) \in (-\infty, t_7] \times \overline{\Omega}. \end{gathered} \tag{2.18}$$

It is easy to check that $(\overline{u}_2, \overline{v}_2)$ and $(\underline{u}_2, \underline{v}_2)$ are the upper and lower solutions of (1.1)–(1.3), and thus from Lemma 2.1, we get

$$\underline{u}_2 \le u \le \overline{u}_2,\ \underline{v}_2 \le v \le \overline{v}_2.$$

From (2.15), (2.17) and (2.18), we have

$$\frac{\partial \underline{v}_2}{\partial t} \ge \Delta \underline{v}_2 + \underline{v}_2 [a_2 - b_2 \underline{v}_2 + \int_0^\infty f_2^-(\tau) \underline{v}_2(t-\tau, x) d\tau - c_2^+(\overline{\beta}_2 + \varepsilon) + d_2(\underline{\alpha}_2 - \varepsilon)].$$

By the comparison principle,

$$\underline{v}_2 \ge \underline{V}_2, \quad t > t_7,\ x \in \Omega,$$

where $\underline{V}_2$ is defined by

$$\begin{gathered} \frac{\partial \underline{V}_2}{\partial t} = \Delta \underline{V}_2 + \underline{V}_2 [a_2 - b_2 \underline{V}_2 + \int_0^\infty f_2^-(\tau) \underline{V}_2(t-\tau, x) d\tau \\ - c_2^+(\overline{\beta}_2 + \varepsilon) + d_2(\underline{\alpha}_2 - \varepsilon)], \quad t > t_7,\ x \in \Omega, \\ \frac{\partial \underline{V}_2}{\partial n} = 0, \quad t > t_7,\ x \in \partial\Omega, \\ \underline{V}_2(t,x) = \frac{1}{2} v(t,x), \quad (t,x) \in (-\infty, t_7] \times \overline{\Omega}. \end{gathered}$$

For sufficiently small ε, we can show that

$$a_2 - c_2^+(\overline{\beta}_2 + \varepsilon) + d_2(\underline{\alpha}_2 - \varepsilon) > 0.$$

From lemma 1.2, we get

$$\lim_{t\to\infty} \underline{V}_2(t,x) = \frac{a_2 - c_2^+ \overline{\beta}_2 + d_2 \underline{\alpha}_2}{b_2 - c_2^-} - \varepsilon \frac{c_2^+ + d_2}{b_2 - c_2^-}, \quad \text{uniformly for } x \in \overline{\Omega}.$$

So for any sufficiently small ε, there exists a $t_8 > t_7$ such that

$$\min_{x\in\overline{\Omega}} \underline{v}_2(t,x) > \underline{\beta}_2 - \varepsilon, \quad t > t_8, \tag{2.19}$$

where $\underline{\beta}_2 = \frac{a_2 - c_2^+ \overline{\beta}_2 + d_2 \underline{\alpha}_2}{b_2 - c_2^-}$. Therefore, for all sufficiently small ε, we can conclude

$$\underline{\alpha}_2 \le \liminf_{t\to\infty} \min_{x\in\overline{\Omega}} u(t,x) \le \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} u(t,x) \le \overline{\alpha}_2, \tag{2.20}$$

$$\underline{\beta}_2 \le \liminf_{t\to\infty} \min_{x\in\overline{\Omega}} v(t,x) \le \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} v(t,x) \le \overline{\beta}_2. \tag{2.21}$$

It is obvious that

$$\underline{\alpha}_1 \le \underline{\alpha}_2 \le \overline{\alpha}_2 \le \overline{\alpha}_1, \quad \underline{\beta}_1 \le \underline{\beta}_2 \le \overline{\beta}_2 \le \overline{\beta}_1 \tag{2.22}$$

Define the sequences $\underline{\alpha}_k$, $\overline{\alpha}_k$, $\underline{\beta}_k$, $\overline{\beta}_k (k \geq 1)$ as follows

$$\begin{aligned} \overline{\alpha}_k &= \frac{a_1 - c_1^+ \underline{\alpha}_{k-1} - d_1 \underline{\beta}_{k-1}}{b_1 - c_1^-}, \overline{\beta}_k = \frac{a_2 - c_2^+ \underline{\beta}_{k-1} + d_2 \overline{\alpha}_k}{b_2 - c_2^-}, \\ \underline{\alpha}_k &= \frac{a_1 - c_1^+ \overline{\alpha}_k - d_1 \overline{\beta}_k}{b_1 - c_1^-}, \quad \underline{\beta}_k = \frac{a_2 - c_2^+ \overline{\beta}_k + d_2 \underline{\alpha}_k}{b_2 - c_2^-}. \end{aligned} \tag{2.23}$$

where $\underline{\alpha}_0 = \underline{\beta}_0 = 0$.

Lemma 2.2. *For the above defined sequences, we have*

$$[\underline{\alpha}_{k+1}, \overline{\alpha}_{k+1}] \subseteq [\underline{\alpha}_k, \overline{\alpha}_k], \quad [\underline{\beta}_{k+1}, \overline{\beta}_{k+1}] \subseteq [\underline{\beta}_k, \overline{\beta}_k], \quad k \geq 1. \tag{2.24}$$

For $k = 1$, it has been shown that $[\underline{\alpha}_2, \overline{\alpha}_2] \subseteq [\underline{\alpha}_1, \overline{\alpha}_1]$, $[\underline{\beta}_2, \overline{\beta}_2] \subseteq [\underline{\beta}_1, \overline{\beta}_1]$. Using induction, we can easily complete the proof, and omit the detail.

Note that Lemma 2.2 implies that the following limits exist: $\lim_{k\to\infty} \underline{\alpha}_k = \underline{\alpha}$, $\lim_{k\to\infty} \overline{\alpha}_k = \underline{\alpha}$, $\lim_{k\to\infty} \underline{\beta}_k = \underline{\beta}$ and $\lim_{k\to\infty} \overline{\beta}_k = \overline{\beta}$. By straightforward computation, we can obtain

$$\begin{aligned} \underline{\alpha} = \overline{\alpha} &= \frac{a_1(b_2 + c_2^+ - c_2^-) - a_2 d_1}{(b_1 + c_1^+ - c_1^-)(b_2 + c_2^+ - c_2^-) + d_1 d_2}, \\ \underline{\beta} = \overline{\beta} &= \frac{a_2(b_1 + c_1^+ - c_1^-) + a_1 d_2}{(b_1 + c_1^+ - c_1^-)(b_2 + c_2^+ - c_2^-) + d_1 d_2}. \end{aligned} \tag{2.25}$$

Lemma 2.3. *For the solutions of* (1.1)–(1.3)*, we have*

$$\underline{\alpha}_k \leq \liminf_{t\to\infty} \min_{x\in\overline{\Omega}} u(t,x) \leq \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} v(t,x) \leq \overline{\alpha}_k, \quad \text{for } k \geq 1, \tag{2.26}$$

$$\underline{\beta}_k \leq \liminf_{t\to\infty} \min_{x\in\overline{\Omega}} u_2(t,x) \leq \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} u_2(t,x) \leq \overline{\beta}_k, \quad \text{for } k \geq 1. \tag{2.27}$$

We have shown that (2.26) and (2.27) are valid for $k = 1, 2$. Using induction and repeating the above process, we can complete the proof of Lemma 2.3.

Combining the above lemmas, we can complete the proof of the main theorem.

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