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DIRICHLET PROBLEMS FOR SEMILINEAR ELLIPTIC EQUATIONS WITH A FAST GROWTH COEFFICIENT ON UNBOUNDED DOMAINS

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Abstract. When an unbounded domain is inside a slab, existence of a positive solution is proved for the Dirichlet problem of a class of semilinear elliptic equations that are similar either to the singular Emden-Fowler equation or a sublinear elliptic equation. The result obtained can be applied to equations with coefficients of the nonlinear term growing exponentially. The proof is based on the super and sub-solution method. A super solution itself is constructed by solving a quasilinear elliptic equation via a modified Perron's method.

1. Introduction and Main Results

Let Ω be an unbounded domain in $\mathbb{R}^n$ $(n \geq 3)$ with $C^{2,\alpha}$ $(0 < \alpha < 1)$ boundary. We assume that Ω is inside a slab of width $2M$:

$$\Omega \subset S_M = \{(\mathbf{x}, y) \in \mathbb{R}^n : |y| < M\}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_{n-1})$ and throughout the paper, y will be identified with x_n. We consider the existence of positive solutions for the Dirichlet problem

$$\begin{gathered} -\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y) D_{ij} u = p(\mathbf{x}, y) u^{\gamma} \quad \text{in } \Omega; \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

where (a_{ij}) is a positive definite matrix in which each entry is a local Hölder continuous function on $\overline{\Omega}$, $p(\mathbf{x}, y)$ is also local Hölder continuous on $\overline{\Omega}$, $\gamma < 1$ is a constant. We note here that (a_{ij}) is not required to be uniformly elliptic on Ω.

When the principal part in (1.1) is the Laplace operator, $\gamma < 0$, (1.1) becomes a boundary-value problem for the singular Emden-Fowler equation

$$\begin{gathered} -\Delta u = p(\mathbf{x}, y) u^{\gamma} \quad \text{in } \Omega; \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

The singular Emden-Fowler is related to the theory of heat conduction in electrical conduction materials and in the studies of boundary layer phenomena for viscous

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fluids ([1], [15]) . The existence of positive solutions of the equation on exterior domains (including $\mathbb{R}^n$) has been widely considered (for example, see [3], [4], [7], [10], [11], [14], and references therein). The main approach used to prove existence results is to construct super and sub solutions. A super solution is usually found in the class of radial symmetric functions. If Ω is an exterior domain (not inside a slab), $\gamma < 0$ and there is C such that $p(\mathbf{x}, y) \geq \frac{C}{(1+|\mathbf{x}|^2+y^2)}$ for $|\mathbf{x}|^2 + y^2$ large, then (1.2) has no positive solutions ([10]). On the other hand, if there are constants $\sigma > 1$ and C, such that $0 \leq p(\mathbf{x}, y) \leq \frac{C}{(1+|\mathbf{x}|^2+y^2)^\sigma}$ for $|\mathbf{x}|^2 + y^2$ large, (1.2) has a positive solution ([7]). When Ω is an unbounded domain inside a slab, the situation is quite different since now one cannot construct a super solution which is a radial symmetric function. In addition, the generality of the coefficient matrix (a_{ij}) in the equation in (1.1) also makes finding a radial symmetric super solution impossible. However a super solution still can be constructed when Ω is a domain inside a slab. In [8], the author combined an idea from [12] and a family of auxiliary functions constructed in [9] to construct a super solution which is then used to prove the following existence result.

Theorem 1.1. *Assume*

1. $p(\mathbf{x}_0, y_0) > 0$ *for some* $(\mathbf{x}_0, y_0) \in \Omega$;
2. *there is a positive constant* C *such that*
$$0 \leq p(\mathbf{x}, y) \leq C(|\mathbf{x}| + 1)^{-\gamma} \quad \text{for} \quad (\mathbf{x}, \mathrm{y}) \in \Omega; \tag{1.3}$$
3. $Trace(a_{ij}) = 1$ *and there is a constant* $c_1 > 0$, *such that*
$$a_{nn}(\mathbf{x}, y) \geq c_1 \quad on \quad \overline{\Omega}. \tag{1.4}$$

Then for $\gamma < 0$, (1.1) *has a positive solution* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.

Comparing Theorem 1 with the known results when the domain is an exterior domain, we see a new phenomena appearing. That is, when the domain is inside a slab, (1.1) has a positive solution even if the coefficient $p(\mathbf{x}, y)$ of the nonlinear term is unbounded, while if the domain is an exterior domain, to assure that (1.1) has a positive solution, the coefficient $p(\mathbf{x}, y)$ of the nonlinear term must go to zero no slower than some functions ([10]). In this paper, we improve Theorem 1 in two aspects. One is to allow the exponent γ in (1.1) to be any number less than 1. The other is to allow the coefficient $p(\mathbf{x}, y)$ of the nonlinear term to grow exponentially! Here is the statement of the main result of the paper.

Theorem 1.2. *Assume* $\gamma < 1$, $\mathrm{Trace}(a_{ij}) = 1$ *and there is a constant* $c_1 > 0$, *such that*
$$a_{nn}(\mathbf{x}, y) \geq c_1 \quad in\ \overline{\Omega}. \tag{1.5}$$
Then there is a positive constant α_0 *depending only on* c_1, M *and* n, *such that for any positive constant* C, (1.1) *has a positive solution* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *for all* $p(\mathbf{x}, y)$ *that is not identical to zero and satisfies*
$$0 \leq p(\mathbf{x}, y) \leq Ce^{(1-\gamma)\alpha_0|\mathbf{x}|} \quad for\ (\mathbf{x}, y) \in \Omega. \tag{1.6}$$
Furthermore, there are constants c_6 *and* c_7 *depending only on* n, M *and* c_1 *such that*
$$u(\mathbf{x}, y) \leq c_6 e^{c_7|\mathbf{x}|} \quad in\ \Omega. \tag{1.7}$$

The idea of the proof of this theorem is as follows: Consider the boundary-value problem

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x},y)D_{ij}u_0 = p(\mathbf{x},y)u_0^{\gamma} \quad \text{in } \Omega; u_0 = 1 \quad \text{on } \partial\Omega. \tag{1.8}$$

For any positive constant $k_0 > 0$, that $u_0 > 0$ satisfies (1.8) is equivalent to that $v_0 = \frac{1}{k_0}\ln u_0$ (i.e. $u_0 = e^{k_0 v_0}$) satisfies

$$\begin{gathered} -\sum_{i,j=1}^{n} a_{ij}(\mathbf{x},y)D_{ij}v_0 = k_0|\nabla v_0|^2 + \frac{1}{k_0}p(\mathbf{x},y)e^{(\gamma-1)k_0v_0} \quad \text{in } \Omega \\ v_0 = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.9}$$

where $|\nabla v_0|^2 = \sum_{i,j=1}^{n} a_{ij}(\mathbf{x},y)D_iv_0D_jv_0$.

We will show that for an appropriately chosen k_0, (1.9) has a positive solution v_0. Then (1.8) has a positive solution $u_0 \geq 1$ that will be a supersolution to (1.1). From there the existence of a positive solution of (1.1) follows from a standard procedure that approximates the solution by solutions on a sequence of bounded domains.

The proof of the existence of a positive solution of (1.9) is very similar to that of the existence of the supersolution in [8]. The main difference is that in [8], a super solution is constructed directly for (1.1) while in this paper, a super solution is constructed through a solution of (1.9). Therefore, the proofs here will be parallel to that in [8] except we need to make some necessary changes to deal with different equations. We give full details of the proofs here so that the paper is self contained and convenient for readers to follow the argument. However for some technical constructions we still refer readers to the paper [8].

2. A Family of Auxiliary Functions

In this section, we use a family of auxiliary functions constructed in [9] to construct families of sub-domains $\Omega_{\mathbf{x}_0}$ of S_M, constants $T_{\mathbf{x}_0}$ and functions $z_{\mathbf{x}_0}$ (see definitions below) such that $T_{\mathbf{x}_0} + z_{\mathbf{x}_0}$ satisfies

$$\begin{aligned} &-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x},y)D_{ij}(T_{\mathbf{x}_0} + z_{\mathbf{x}_0}) \\ &\geq k_0|\nabla(T_{\mathbf{x}_0} + z_{\mathbf{x}_0})|^2 + \frac{1}{k_0}p(\mathbf{x},y)e^{(\gamma-1)k_0(T_{\mathbf{x}_0}+z_{\mathbf{x}_0})} \quad \text{in } \Omega_{\mathbf{x}_0}\cap\Omega \end{aligned} \tag{2.1}$$

and the graphs of the functions $T_{\mathbf{x}_0} + z_{\mathbf{x}_0}$ have special relative positions (see below).

We first extend a_{ij} $(1 \leq i,j \leq n)$ to be continuous functions on $\overline{S_M}$ in such a way that we still have $Trace(a_{ij}) = 1$ and

$$a_{nn}(\mathbf{x},y) \geq c_1 \quad \text{in } S_M. \tag{2.2}$$

It was proved in [9] (the construction of the functions was inspired by [13] and the details of the construction were also repeated in the appendix in [8]) that there are positive decreasing functions $\chi(t)$, $h_a(t)$ and a positive increasing function $A(t)$ ($\chi(t)$ depending on c_1 only, $h_a(t)$ and $A(t)$ depending on c_1 and M only), such that for any number K, there is a number H_0, depending only on K, M and c_1, such that for $H \ge H_0$, we have (for $0 < t < 2M$)

$$A(H) \le h_a^{-1}(t) \le A(H)e^{\chi(H)}, \quad 22MH \le c_1 A(H)e^{\chi(H)} \le 66MH, \tag{2.3}$$

$$8K \le A(H)e^{\chi(H)}, \quad 0 < \chi(H) < 1, \tag{2.4}$$

and the non-negative function

$$z = z_{\mathbf{x}_0} = A(H)e^{\chi(H)} - \{(h_a^{-1}(y+M))^2 - |\mathbf{x}-\mathbf{x}_0|^2\}^{1/2} \tag{2.5}$$

satisfies

$$\sum_{i,j=1}^{n} a_{ij}(\mathbf{x},y)D_{ij}z \le \frac{-3c_1}{22eMH} \quad \text{in } \Omega_{\mathbf{x}_0,H,K}, \tag{2.6}$$

$$z \ge K \quad \text{on } \partial\Omega_{\mathbf{x}_0,H,K} \cap \{|y| < M\}, \quad z(\mathbf{x}_0,y) \le \frac{2M}{H} \quad \text{for } |y| \le M, \tag{2.7}$$

$$|D_{\mathbf{x}}z(\mathbf{x},y)| \le 2(\frac{c_1K}{M})^{1/2}\frac{1}{\sqrt{H}}, \quad |D_y z(\mathbf{x},y)| \le \frac{2}{H} \quad \text{on } \Omega_{\mathbf{x}_0,H,K}, \tag{2.8}$$

where

$$\Omega_{\mathbf{x}_0,H,K} = \big\{(\mathbf{x},y) : |y| < M, |\mathbf{x}-\mathbf{x}_0| < \sqrt{\frac{2K}{A(H)e^{\chi(H)}}}h_a^{-1}(y+M)\big\}. \tag{2.9}$$

Now we set

$$K = 100, \quad H = H_0 + 4M, \quad c_2 = \frac{3c_1}{22eMH}, \quad \Omega_{\mathbf{x}_0} = \Omega_{\mathbf{x}_0,H,K}. \tag{2.10}$$

Then from (2.7), we have

$$z \ge 100 \quad \text{on } \partial\Omega_{\mathbf{x}_0} \cap \{|y| < M\}, \quad z(\mathbf{x}_0,y) \le 1 \quad \text{for } |y| \le M. \tag{2.11}$$

For two points $\mathbf{x}_0$ and $\mathbf{x}_1$ in R^{n-1}, when $\Omega_{\mathbf{x}_1}$ either covers the whole segment of the set $\{(\mathbf{x}_0,y)||y| \le M\}$ or does not intersect with the set, from (2.3) and (2.9), we have either

$$|\mathbf{x}_1-\mathbf{x}_0| \le \sqrt{200A(H)e^{-\chi(H)}} \quad \text{or} \quad |\mathbf{x}_1-\mathbf{x}_0| \ge \sqrt{200A(H)e^{\chi(H)}}. \tag{2.12}$$

When $\Omega_{\mathbf{x}_1}$ covers part of the set $\{(\mathbf{x}_0,y) : |y| \le M\}$, we have

$$\sqrt{195A(H)e^{-\chi(H)}} \le |\mathbf{x}_1-\mathbf{x}_0| \le \sqrt{205A(H)e^{\chi(H)}}. \tag{2.13}$$

Let $\mathbf{x}_1$ and $\mathbf{x}_0$ satisfy (2.13) and δ_0 be a small positive number such that $2\delta_0 < \sqrt{195A(H)e^{-\chi(H)}}$. If $(\mathbf{x},y) \in \Omega_{\mathbf{x}_1}$ for some y and $|\mathbf{x}-\mathbf{x}_0| \le \delta_0$, by (2.3), (2.5) and (2.13), we have

$$\begin{aligned}
&z_{\mathbf{x}_1}(\mathbf{x},y)\\
&\ge A(H)e^{\chi(H)} - \{A(H)^2e^{2\chi(H)} - |\mathbf{x}-\mathbf{x}_1|\}^{1/2}\\
&\ge A(H)e^{\chi(H)} - \{A(H)^2e^{2\chi(H)} - (\sqrt{195A(H)e^{-\chi(H)}} - \delta_0)^2\}^{1/2}\\
&\ge A(H)e^{\chi(H)} - \{A(H)^2e^{2\chi(H)} - 195A(H)e^{-\chi(H)} + 2\delta_0\sqrt{195A(H)e^{-\chi(H)}}\}^{1/2}\\
&\ge A(H)e^{\chi(H)}(1-(1-\frac{195}{A(H)e^{3\chi(H)}} + \frac{2\delta_0\sqrt{195A(H)e^{-\chi(H)}}}{A(H)^2e^{2\chi(H)}})^{1/2})
\end{aligned}$$

(by the inequality $\sqrt{1-t} \le 1 - \frac{1}{2}t$ for $0 < t < 1$ and (2.4))

$$\begin{aligned} &\ge A(H)e^{\chi(H)}\Big(\frac{195}{2A(H)e^{3\chi(H)}} - \frac{2\delta_0\sqrt{195A(H)e^{-\chi(H)}}}{2A(H)^2e^{2\chi(H)}}\Big) \\ &= \frac{195}{2e^{2\chi(H)}} - \frac{\delta_0\sqrt{195A(H)e^{-\chi(H)}}}{A(H)e^{\chi(H)}} > 10 - \frac{\delta_0\sqrt{195A(H)e^{-\chi(H)}}}{A(H)e^{\chi(H)}}. \end{aligned}$$

Thus there is a δ_0 small such that for all $|\mathbf{x} - \mathbf{x}_0| \le \delta_0$ with $(\mathbf{x}, y) \in \Omega_{\mathbf{x}_1}$, if $\mathbf{x}_1$ and $\mathbf{x}_0$ satisfy (2.13), we have

$$z_{\mathbf{x}_1}(\mathbf{x}, y) \ge 8. \tag{2.14}$$

From (2.9) and (2.11), we can choose a number $\delta_2(\mathbf{x}_0) > 0$ such that for all $\mathbf{x} \in R^{n-1}$ with $|\mathbf{x}_0 - \mathbf{x}| \le \delta_2(\mathbf{x}_0)$, we have $(\mathbf{x}, y) \in \Omega_{\mathbf{x}_0}$ for all $|y| < M$, and

$$z_{\mathbf{x}_0}(\mathbf{x}, y) \le 2. \tag{2.15}$$

Now if we set $\delta_{\mathbf{x}_0} = \min\{\delta_0, \delta_2(\mathbf{x}_0)\}$, from (2.14) and (2.15), we have

$$z_{\mathbf{x}_0}(\mathbf{x}, y) \le 2 < 8 \le z_{\mathbf{x}_1}(\mathbf{x}, y) \tag{2.16}$$

for all $\mathbf{x}_0$ and $\mathbf{x}_1$ satisfying (2.13), $|\mathbf{x}_0 - \mathbf{x}| \le \delta_{\mathbf{x}_0}$ and $(\mathbf{x}, y) \in \Omega_{\mathbf{x}_1}$.

Since $\mathrm{Trace}(a_{ij}) = 1$, we have $\sum a_{ij}\xi_i\xi_j \le 1$ for any unit vector $\xi = (\xi_1, \dots, \xi_n)$. Then from (2.8), we have

$$\begin{aligned} |\nabla z|^2 &= \sum a_{ij}z_iz_j \le |D_{\mathbf{x}}z|^2 + |D_yz|^2 \\ &\le \frac{400c_1}{MH} + \frac{4}{H^2} \\ &\le \frac{400c_1}{MH} + \frac{1}{HM} = \frac{400c_1 + 1}{MH} = c_3\,. \end{aligned} \tag{2.17}$$

Here we have used that $H > 4M$. If we set $k_0 = \frac{c_2}{2c_3}$, then the function $z_{\mathbf{x}_0}$ satisfies that on $\Omega_{\mathbf{x}_0}$

$$k_0|\nabla z|^2 \le \frac{c_2}{2c_3}c_3 = \frac{1}{2}c_2. \tag{2.18}$$

Now we set $c_4 = \sqrt{205A(H)e^{\chi(H)}}$, $c_5 = \sqrt{200A(H)e^{\chi(H)}}$, $\alpha_0 = \frac{1}{c_4}k_0$ and assume that for some constant C, $p(\mathbf{x}, y)$ satisfies (1.6). Furthermore we set

$$T_{\mathbf{x}_0} = \frac{1}{c_4}(|\mathbf{x}_0| + A)$$

where $A = c_5 - \frac{c_4}{(1-\gamma)k_0}\ln(\min\{\frac{c_2k_0}{2C}, 1\})$. Since on $\Omega_{\mathbf{x}_0}$, $|\mathbf{x}| \le |\mathbf{x}_0| + c_5$, from (1.6), we have that on $\Omega_{\mathbf{x}_0} \cap \Omega$,

$$\begin{aligned} \frac{1}{k_0}p(\mathbf{x}, y)e^{(\gamma-1)k_0(T_{\mathbf{x}_0}+z)} &\le \frac{C}{k_0}e^{\frac{1}{c_4}k_0(1-\gamma)|\mathbf{x}|}e^{(\gamma-1)k_0T_{\mathbf{x}_0}} \\ &\le \frac{C}{k_0}e^{\frac{1}{c_4}k_0(1-\gamma)(|\mathbf{x}_0|+c_5)}e^{(\gamma-1)k_0\frac{1}{c_4}(|\mathbf{x}_0|+A)} \\ &= \frac{C}{k_0}e^{\frac{1}{c_4}k_0(1-\gamma)c_5}e^{(\gamma-1)k_0\frac{1}{c_4}A} \le \frac{1}{2}c_2 \end{aligned} \tag{2.19}$$

by the definition of A. Combining (2.6), (2.18) and (2.19), we have that on $\Omega_{\mathbf{x}_0} \cap \Omega$, $T_{\mathbf{x}_0} + z$ satisfies

$$k_0|\nabla(T_{\mathbf{x}_0} + z)|^2 + \frac{1}{k_0}p(\mathbf{x}, y)e^{(\gamma-1)k_0(T_{\mathbf{x}_0}+z)} \le c_2 \le -\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y)D_{ij}(T_{\mathbf{x}_0} + z)\,.$$

That is, $T_{\mathbf{x}_0} + z_{\mathbf{x}_0}$ satisfies (2.1). As $\mathbf{x}_0$ varies, we get a family of such functions.

Now for all $\mathbf{x}_0$ and $\mathbf{x}_1$ satisfying (2.13),

$$\begin{aligned} T_{\mathbf{x}_0} &= T_{\mathbf{x}_1} + T_{\mathbf{x}_0} - T_{\mathbf{x}_1} \\ &= T_{\mathbf{x}_1} + \frac{1}{c_4}(|\mathbf{x}_0| - |\mathbf{x}_1|) \\ &\le T_{\mathbf{x}_1} + \frac{1}{c_4}|\mathbf{x}_0 - \mathbf{x}_1| \le T_{\mathbf{x}_1} + 1. \end{aligned}$$

Then for all $\mathbf{x}_0$ and $\mathbf{x}_1$ satisfying (2.13), $|\mathbf{x}_0 - \mathbf{x}| \le \delta_{\mathbf{x}_0}$ and $(\mathbf{x}, y) \in \Omega_{\mathbf{x}_1}$, from (2.16), we have

$$T_{\mathbf{x}_0} + z_{\mathbf{x}_0}(\mathbf{x}, y) \le T_{\mathbf{x}_1} + 1 + 2 < T_{\mathbf{x}_1} + z_{\mathbf{x}_0}(\mathbf{x}, y). \tag{2.20}$$

Finally we define a family of open subsets of Ω that will be needed in next section.

For each point $(\mathbf{x}_0, y_0) \in \overline{\Omega}$, we define an open set $O(\mathbf{x}_0, y_0)$ as follows:

1) If $(\mathbf{x}_0, y_0) \in \Omega$, we choose a ball B with center $(\mathbf{x}_0, y_0)$ and a radius less than $\delta_{\mathbf{x}_0}$ so that $B \subset \Omega$. We then set $O(\mathbf{x}_0, y_0) = B$;

2) If $(\mathbf{x}_0, y_0) \in \partial\Omega$, since Ω has $C^{2,\alpha}$ boundary, there is a ball B with center $(\mathbf{x}_0, y_0)$ and a radius less than $\delta_{\mathbf{x}_0}$, such that there is a $C^{2,\alpha}$ diffeomorphism Φ that satisfies

$$\Phi(B \cap \Omega) \subset \mathbb{R}^n_+, \quad \Phi(B \cap \partial\Omega) \subset \partial\mathbb{R}^n_+; \quad \Phi(\mathbf{x}_0, y_0) = \mathbf{0}.$$

Now we choose a domain J with C^3 boundary with following properties: (a) $J \subset \Phi(B \cap \Omega)$; (b) $\partial J \cap \partial\mathbb{R}^n_+$ is a neighborhood of $\mathbf{0}$ in $\partial\mathbb{R}^n_+$. Certainly there are many different J's having those properties. One example on how to construct J is given in the Appendix II [8].

Now we set $O(\mathbf{x}_0, y_0) = \Phi^{-1}(J)$. It is easy to see that $O(\mathbf{x}_0, y_0) \subset B \cap \Omega$, $O(\mathbf{x}_0, y_0)$ has a $C^{2,\alpha}$ boundary and $\partial O(\mathbf{x}_0, y_0) \cap \partial\Omega$ is a neighborhood of $(\mathbf{x}_0, y_0)$ in $\partial\Omega$.

Let Π be the collection of all such open sets $O(\mathbf{x}_0, y_0)$ defined in 1) and 2).

3. A Solution of (1.9)

In [12], a modified version of the Perron's method has been used to prove the existence of solutions. In the modified version of the Perron's method, one uses a family of local upper barriers to replace the role played by the supersolution in the normal version of the Perron's method (the local upper barriers in [12] were inspired by [5]). The modified version has been used in [8] to prove the existence of a super solution by using a family of auxiliary functions similar to the one constructed in section 2. In this section, we will follow the same part in [8] with the necessary modifications. We will show that there is a positive function $v_0 \in C^2(\Omega) \cap C^0(\overline{\Omega})$, satisfies

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y) D_{ij} v_0 = k_0 |\nabla v_0|^2 + \frac{1}{k_0} p(\mathbf{x}, y) e^{(\gamma-1)k_0 v_0} \quad \text{in } \Omega, \tag{3.1}$$

$$v_0 = 0 \quad \text{on } \partial\Omega. \tag{3.2}$$

Let $v \ge 0$ be a continuous function on $\overline{\Omega}$, for a point $(\mathbf{x}_0, y_0) \in \overline{\Omega}$, we define a new function $M_{(\mathbf{x}_0,y_0)}(v)$, called the lift of v over $O(\mathbf{x}_0, y_0)$ as follows:

$$M_{(\mathbf{x}_0,y_0)}(v)(\mathbf{x}, y) = \begin{cases} v(\mathbf{x}, y) & \text{if } (\mathbf{x}, y) \in \Omega \setminus O(\mathbf{x}_0, y_0) \\ w(\mathbf{x}, y) & \text{if } (\mathbf{x}, y) \in O(\mathbf{x}_0, y_0) \end{cases}$$

where $w(\mathbf{x}, y)$ is the non-negative solution of the boundary-value problem

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y)D_{ij}w = k_0|\nabla w|^2 + \frac{1}{k_0}p(\mathbf{x}, y)e^{(\gamma-1)k_0 w} \quad \text{in } O(\mathbf{x}_0, y_0), \tag{3.3}$$

$$w = v \quad \text{on } \partial O(\mathbf{x}_0, y_0). \tag{3.4}$$

We claim that this system has a solution in $C^2(O(\mathbf{x}_0, y_0)) \cap C^0(\overline{O(\mathbf{x}_0, y_0)})$, which is positive and unique. Indeed the uniqueness and positivity of a solution easily follow from a standard maximum principle. For existence, we notice that $m_1 = \min\{v(\mathbf{x}, y)|(\mathbf{x}, y) \in \partial O(\mathbf{x}_0, y_0)\}$ is a sub-solution, $m_2 + T_{\mathbf{x}_0} + z_{\mathbf{x}_0}$ is a super solution by (2.1), where $m_2 = \max\{v(\mathbf{x}, y)|(\mathbf{x}, y) \in \partial O(\mathbf{x}_0, y_0)\}$. We set a change of variable

$$u = e^{k_0 w}.$$

Then w satisfies (3.3)-(3.4) if and only if u satisfies

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y)D_{ij}u = p(\mathbf{x}, y)u^{\gamma} \quad \text{in } O(\mathbf{x}_0, y_0), \tag{3.5}$$

$$u = e^{k_0 w} \quad \text{on } \partial O(\mathbf{x}_0, y_0). \tag{3.6}$$

Then $e^{k_0 m_1}$ and $e^{k_0(m_2 + T_{\mathbf{x}_0} + z_{\mathbf{x}_0})}$ are sub- and super- solutions of (3.5)-(3.6). It is then well known that (3.5)-(3.6) has a positive solution u (see [2]). Then $w = \frac{1}{k_0}\ln u$ solves (3.3)-(3.4).

We define a class Ξ of functions as follows: A function v is in Ξ if

(1) $v \in C^0(\overline{\Omega})$, $v \geq 0$ on Ω and $v \leq 0$ on $\partial\Omega$;
(2) For any $(\mathbf{x}_0, y_0) \in \overline{\Omega}$, $v \leq M_{(\mathbf{x}_0, y_0)}(v)$;
(3) $v \leq T_{\mathbf{x}_0} + z_{\mathbf{x}_0}$ on $\Omega_{\mathbf{x}_0} \cap \Omega$ for any $(\mathbf{x}_0, y_0) \in \overline{\Omega}$.

An application of a maximum principle implies that the function $v = 0$ is in Ξ. Thus Ξ is not empty. Now we set

$$v_0(\mathbf{x}, y) = \sup_{v \in \Xi} v(\mathbf{x}, y), \quad (\mathbf{x}, y) \in \overline{\Omega}.$$

We will show that v_0 is positive on Ω, in $C^2(\Omega) \cap C^0(\overline{\Omega})$ and satisfies (3.1)-(3.2). First we present some lemmas.

Lemma 3.1. *Let D be a bounded domain, If w_1, w_2 are in $C^2(D) \cap C^0(\overline{D})$, $w_1 \leq w_2$ on ∂D, and*

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y)D_{ij}w_1 \leq k_0|\nabla w_1|^2 + \frac{1}{k_0}p(\mathbf{x}, y)e^{(\gamma-1)k_0 w_1} \quad \text{in } D,$$

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y)D_{ij}w_2 \geq k_0|\nabla w_2|^2 + \frac{1}{k_0}p(\mathbf{x}, y)e^{(\gamma-1)k_0 w_2} \quad \text{in } D$$

then $w_1 \leq w_2$ on D.

Since $\frac{1}{k_0}p(\mathbf{x}, y)e^{(\gamma-1)k_0 t}$ is decreasing on t, a straightforward application of a maximum principle to $w_1 - w_2$ gives the proof of the above lemma.

Lemma 3.2. *If $0 < v_1 \leq v_2$, then $M_{(\mathbf{x}_0, y_0)}(v_1) \leq M_{(\mathbf{x}_0, y_0)}(v_2)$ for any $(\mathbf{x}_0, y_0) \in \overline{\Omega}$.*

Proof. Let w_1, w_2 be the positive solutions for the following problems, respectively,

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x},y)D_{ij}w_q = k_0|\nabla w_q|^2 + \frac{1}{k_0}p(\mathbf{x},y)e^{(\gamma-1)k_0 w_q} \quad \text{in } O(\mathbf{x}_0,y_0),$$
$$w_q = v_q \quad \text{on } \partial O(\mathbf{x}_0,y_0), \quad q=1,2.$$

Since $w_1 = v_1 \le v_2 = w_2$ on $\partial O(\mathbf{x}_0,y_0)$, from lemma 1, we see $w_1 \le w_2$ on $O(\mathbf{x}_0,y_0)$. However, on $\Omega \setminus O(\mathbf{x}_0,y_0)$, $M_{(\mathbf{x}_0,y_0)}(v_1) = v_1$, $M_{(\mathbf{x}_0,y_0)}(v_2) = v_2$. Thus $M_{(\mathbf{x}_0,y_0)}(v_1) \le M_{(\mathbf{x}_0,y_0)}(v_2)$. □

Lemma 3.3. *If* $v_1 \in \Xi$, $v_2 \in \Xi$, *then* $\max\{v_1,v_2\} \in \Xi$.

Proof. If $v_1 \in \Xi$, $v_2 \in \Xi$, it is clear that $\max\{v_1,v_2\} \in C^0(\overline{\Omega})$, $\max\{v_1,v_2\} \ge 0$ on Ω and $\max\{v_1,v_2\} \le 0$ on $\partial\Omega$. It is also clear that $\max\{v_1,v_2\} \le T_{\mathbf{x}_0} + z_{\mathbf{x}_0}$ on $\Omega_{\mathbf{x}_0} \cap \Omega$ for any $(\mathbf{x}_0,y_0) \in \overline{\Omega}$. Since

$$v_1 \le \max\{v_1,v_2\}, \quad v_2 \le \max\{v_1,v_2\},$$

by lemma 2 we have that for any $(\mathbf{x}_0,y_0) \in \overline{\Omega}$,

$$M_{(\mathbf{x}_0,y_0)}(v_1) \le M_{(\mathbf{x}_0,y_0)}(\max\{v_1,v_2\}),$$
$$M_{(\mathbf{x}_0,y_0)}(v_2) \le M_{(\mathbf{x}_0,y_0)}(\max\{v_1,v_2\}).$$

Since $v_1 \in \Xi$ and $v_2 \in \Xi$ imply

$$v_1 \le M_{(\mathbf{x}_0,y_0)}(v_1), \quad \text{and} \quad v_2 \le M_{(\mathbf{x}_0,y_0)}(v_2),$$

we have

$$\max\{v_1,v_2\} \le M_{(\mathbf{x}_0,y_0)}(\max\{v_1,v_2\}).$$

Thus $\max\{v_1,v_2\} \in \Xi$. □

Lemma 3.4. *If* $v \in \Xi$, *then* $M_{(\mathbf{x}_0,y_0)}(v) \in \Xi$ *for any* $(\mathbf{x}_0,y_0) \in \overline{\Omega}$.

Proof. By the definition of $M_{(\mathbf{x}_0,y_0)}(v)$, it is clear that $M_{(\mathbf{x}_0,y_0)}(v) \ge 0$ on Ω, $M_{(\mathbf{x}_0,y_0)}(v) \in C^0(\overline{\Omega})$ and $M_{(\mathbf{x}_0,y_0)}(v) \le 0$ on $\partial\Omega$. For any $(\mathbf{x}^*,y^*) \in \overline{\Omega}$, we first show that

$$M_{(\mathbf{x}_0,y_0)}(v)(\mathbf{x},y) \le M_{(\mathbf{x}^*,y^*)}(M_{(\mathbf{x}_0,y_0)}(v))(\mathbf{x},y). \tag{3.7}$$

We only need to prove (3.7) for $(\mathbf{x},y) \in O(\mathbf{x}^*,y^*)$.

Since $v \le M_{(\mathbf{x}_0,y_0)}(v)$, by lemma 2 we have

$$M_{(\mathbf{x}^*,y^*)}(v) \le M_{(\mathbf{x}^*,y^*)}(M_{(\mathbf{x}_0,y_0)}(v)).$$

Then from $v \le M_{(\mathbf{x}^*,y^*)}(v)$, we have

$$v \le M_{(\mathbf{x}^*,y^*)}(M_{(\mathbf{x}_0,y_0)}(v)).$$

Thus for $(\mathbf{x},y) \in O(\mathbf{x}^*,y^*) \setminus O(\mathbf{x}_0,y_0)$,

$$M_{(\mathbf{x}_0,y_0)}(v)(\mathbf{x},y) = v(\mathbf{x},y) \le M_{(\mathbf{x}^*,y^*)}(M_{(\mathbf{x}_0,y_0)}(v))(\mathbf{x},y). \tag{3.8}$$

That is, (3.7) is true on $O(\mathbf{x}^*,y^*) \setminus O(\mathbf{x}_0,y_0)$, Now for $\Omega_1 = O(\mathbf{x}^*,y^*) \cap O(\mathbf{x}_0,y_0)$, if we set

$$M_{(\mathbf{x}_0,y_0)}(v) = w_1, \quad M_{(\mathbf{x}^*,y^*)}(M_{(\mathbf{x}_0,y_0)}(v)) = w_2$$

then

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x},y)D_{ij}w_1 = k_0|\nabla w_1|^2 + \frac{1}{k_0}p(\mathbf{x},y)e^{(\gamma-1)k_0 w_1} \quad \text{on } \Omega_1,$$
$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x},y)D_{ij}w_2 = k_0|\nabla w_2|^2 + \frac{1}{k_0}p(\mathbf{x},y)e^{(\gamma-1)k_0 w_2} \quad \text{on } \Omega_1.$$

On $\partial\Omega_1$, $w_1 \le w_2$ on $O(\mathbf{x}^*, y^*) \cap \partial O(\mathbf{x}_0, y_0)$ by (3.8) and $w_1 \le w_2$ on $\partial O(\mathbf{x}^*, y^*) \cap O(\mathbf{x}_0, y_0)$ since (3.7) is true on $\Omega \setminus O(\mathbf{x}^*, y^*)$. Then lemma 1 implies $w_1 \le w_2$ on Ω_1. Thus (3.7) is true on $O(\mathbf{x}^*, y^*) \cap O(\mathbf{x}_0, y_0)$ and on $O(\mathbf{x}^*, y^*)$.

Now we prove that $M_{(\mathbf{x}_0,y_0)}(v) \le T_{\mathbf{x}_1} + z_{\mathbf{x}_1}$ on $\Omega_{\mathbf{x}_1} \cap \Omega$ for all $(\mathbf{x}_1, y_1) \in \overline{\Omega}$. By the definition of $M_{(\mathbf{x}_0,y_0)}(v)$, we only need to consider the graph of the function $M_{(\mathbf{x}_0,y_0)}(v)$ over $O(\mathbf{x}_0, y_0)$. If $O(\mathbf{x}_0, y_0)$ is covered completely by $\Omega_{\mathbf{x}_1}$, since $v \le T_{\mathbf{x}_1} + z_{\mathbf{x}_1}$ and $T_{\mathbf{x}_1} + z_{\mathbf{x}_1}$ satisfies (2.1), $T_{\mathbf{x}_1} + z_{\mathbf{x}_1}$ is a super solution of (3.3) on $O(\mathbf{x}_0, y_0)$. Then Lemma 1 implies $M_{(\mathbf{x}_0,y_0)}(v) \le T_{\mathbf{x}_1} + z_{\mathbf{x}_1}$ on $O(\mathbf{x}_0, y_0)$. In the case that $O(\mathbf{x}_0, y_0)$ does not intersect with $\Omega_{\mathbf{x}_1}$, the conclusion is trivial. Now we consider the case that $O(\mathbf{x}_0, y_0)$ is partially covered by $\Omega_{\mathbf{x}_1}$. First by what we have just proved, we always have

$$M_{(\mathbf{x}_0,y_0)}(v) \le T_{\mathbf{x}_0} + z_{\mathbf{x}_0} \quad \text{on } O(\mathbf{x}_0, y_0). \tag{3.9}$$

By the choice of $\delta_{\mathbf{x}_0}$ and $O(\mathbf{x}_0, y_0)$, the graph of $T_{\mathbf{x}_0} + z_{\mathbf{x}_0}$ over $O(\mathbf{x}_0, y_0) \cap \Omega_{\mathbf{x}_1}$ is under the graph of $T_{\mathbf{x}_1} + z_{\mathbf{x}_1}$. Thus the conclusion follows from (3.9). □

Now we are ready to prove that v_0 has the desired properties. Let $(\mathbf{x}_0, y_0) \in \overline{\Omega}$. By the definition of $v_0(\mathbf{x}_0, y_0)$, there is a sequence of functions v_k in Ξ such that

$$v_0(\mathbf{x}_0, y_0) = \lim_{k\to\infty} v_k(\mathbf{x}_0, y_0).$$

By the definition of Ξ, $v_k \ge 0$ on Ω. We replace v_k by $M_{(\mathbf{x}_0,y_0)}(v_k)$. Then we have a sequence of functions w_k such that

$$v_0(\mathbf{x}_0, y_0) = \lim_{k\to\infty} w_k(\mathbf{x}_0, y_0),$$
$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x},y)D_{ij}w_k = k_0|\nabla w_k|^2 + \frac{1}{k_0}p(\mathbf{x},y)e^{(\gamma-1)k_0 w_k} \quad \text{on } O(\mathbf{x}_0, y_0),$$
$$w_k = v_k \quad \text{on } \partial O(\mathbf{x}_0, y_0).$$

Then $u_k = e^{k_0 w_k}$ satisfies

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x},y)D_{ij}u_k = p(\mathbf{x},y)u_k^{\gamma} \quad \text{on } O(\mathbf{x}_0, y_0),$$
$$u_k = e^{k_0 v_k} \quad \text{on } \partial O(\mathbf{x}_0, y_0).$$

Further, from the fact that for all k,

$$0 \le v_k \le w_k \le T_{\mathbf{x}_0} + z_{\mathbf{x}_0} \quad \text{on } O(\mathbf{x}_0, y_0),$$

we have

$$1 \le u_k \le e^{k_0(T_{\mathbf{x}_0} + z_{\mathbf{x}_0})} \quad \text{on } O(\mathbf{x}_0, y_0).$$

By [6, Theorem 9.11] and an approximation of the boundary value by smooth functions, we see that there is a subsequence of u_k, for convenience still denoted by

u_k, converges to a $C^2(O(\mathbf{x}_0,y_0))\cap C^0(\overline{O(\mathbf{x}_0,y_0)})$ function $u_*(x)$ in $C^2(O(\mathbf{x}_0,y_0))\cap C^0(\overline{O(\mathbf{x}_0,y_0)})$. Thus $u_*(x)$ satisfies

$$-\sum_{i,j=1}^n a_{ij}(\mathbf{x},y)D_{ij}u_* = p(\mathbf{x},y)u_*^\gamma \quad \text{on } O(\mathbf{x}_0,y_0)$$

Then $w=\frac{1}{k_0}\ln u_*$ satisfies

$$-\sum_{i,j=1}^n a_{ij}(\mathbf{x},y)D_{ij}w = k_0|\nabla w|^2 + \frac{1}{k_0}p(\mathbf{x},y)e^{(\gamma-1)k_0 w} \quad \text{on } O(\mathbf{x}_0,y_0)$$

and $v_0(\mathbf{x}_0,y_0)=w(\mathbf{x}_0,y_0)$. We claim that $v_0=w$ on $O(\mathbf{x}_0,y_0)$. Indeed, if there is another point $(\mathbf{x}_2,y_2)\in O(\mathbf{x}_0,y_0)$ such that $v_0(\mathbf{x}_2,y_2)$ is not equal to $w(\mathbf{x}_2,y_2)$, then $v_0(\mathbf{x}_2,y_2)>w(\mathbf{x}_2,y_2)$. Then there is a function $v^*\in\Xi$, such that

$$w(\mathbf{x}_2,y_2)<v^*(\mathbf{x}_2,y_2)\le v_0(\mathbf{x}_2,y_2).$$

Now the sequence $\max\{v^*,M_{(\mathbf{x}_0,y_0)}(v_k)\}$ satisfies

$$v_k\le \max\{v^*,M_{(\mathbf{x}_0,y_0)}(v_k)\}\le v_0.$$

In a similar way, $M_{(\mathbf{x}_0,y_0)}(\max\{v^*,M_{(\mathbf{x}_0,y_0)}(v_k)\})$ will produce a function w_1 such that

$$\begin{gathered}-\sum_{i,j=1}^n a_{ij}(\mathbf{x},y)D_{ij}w_1 = k_0|\nabla w_1|^2 + \frac{1}{k_0}p(\mathbf{x},y)e^{(\gamma-1)k_0 w_1} \quad \text{on } O(\mathbf{x}_0,y_0),\\ w\le w_1 \quad \text{on } O(\mathbf{x}_0,y_0),\\ w(\mathbf{x}_2,y_2)<v^*(\mathbf{x}_2,y_2)\le w_1(\mathbf{x}_2,y_2),\\ w(\mathbf{x}_0,y_0)=w_1(\mathbf{x}_0,y_0)=v_0(\mathbf{x}_0,y_0).\end{gathered}$$

That is, $w_1(\mathbf{x},y)-w(\mathbf{x},y)$ is non-negative, not identically zero on $O(\mathbf{x}_0,y_0)$ and achieves its minimum value zero inside $O(\mathbf{x}_0,y_0)$. However, from the equations satisfied by w and w_1, we have that on $O(\mathbf{x}_0,y_0)$,

$$\begin{gathered}-\sum_{i,j=1}^n a_{i,j}(\mathbf{x},y)D_{ij}(w_1-w) - 2k_0(\nabla w+\theta_1\nabla(w_1-w))\cdot\nabla(w_1-w)\\ -(\gamma-1)p(\mathbf{x},y)e^{(\gamma-1)k_0(w+\theta_2(w-w_1))}(w-w_1)=0\end{gathered}$$

for some continuous functions θ_1 and θ_2. Then by the standard maximum principle (see [6, Theorem 3.5]), we get a contradiction. Thus $v_0=w$ on $O(\mathbf{x}_0,y_0)$. Therefore $v_0\in C^2(\Omega)$ and

$$-\sum_{i,j=1}^n a_{i,j}(\mathbf{x},y)D_{ij}v_0 = k_0|\nabla v_0|^2 + \frac{1}{k_0}p(\mathbf{x},y)e^{(\gamma-1)k_0 v_0}.$$

When $(\mathbf{x}_0,y_0)\in\partial\Omega$, $\partial O(\mathbf{x}_0,y_0)\cap\partial\Omega$ is a neighborhood of $(\mathbf{x}_0,y_0)$ in $\partial\Omega$. Since $\max\{0,v_k\}=0$ on $\partial\Omega$, $v_0=0$ on $\partial\Omega$ and $w=0$ on $\partial O(\mathbf{x}_0,y_0)\cap\partial\Omega$. Since w is continuous up to the boundary of $O(\mathbf{x}_0,y_0)$, v_0 is continuous on $\partial O(\mathbf{x}_0,y_0)\cap\partial\Omega$ from inside $O(\mathbf{x}_0,y_0)$. Thus $v_0\in C^0(\overline{\Omega})$ and $v_0=0$ on $\partial\Omega$. Now from the definition of $T_{\mathbf{x}_0}$ and $v\le T_{\mathbf{x}_0}+z_{\mathbf{x}_0}$ on $\Omega_{\mathbf{x}_0}$ for all $v\in\Xi$, we have

$$v_0(\mathbf{x}_0,y)\le T_{\mathbf{x}_0}+z_{\mathbf{x}_0}(\mathbf{x}_0,y)\le\frac{1}{c_4}|\mathbf{x}_0|+A+3$$

for all $(\mathbf{x}_0, y) \in \Omega$. If we let $\mathbf{x}_0$ vary, we get

$$v_0(\mathbf{x}, y) \le \frac{1}{c_4}|\mathbf{x}| + A + 3 \quad \text{on } \Omega. \tag{3.10}$$

Then $u_0 = e^{k_0 v_0}$ satisfies (1.8), $u_0 \ge 1$ and $u_0 \le c_6 e^{c_7|\mathbf{x}|}$ for some constants c_6 and c_7 depending only on n, M and c_1.

4. The Proof of Theorem 1

Since Ω is an unbounded domain with $C^{2,\alpha}$ boundary, we can choose a sequence of subdomains in Ω, denoted by Ω_m, $m = 1, 2, 3, \dots$, such that

(1) $\Omega_m \subset \Omega_{m+1} \subset \Omega$ for all m;
(2) $\cup \Omega_m = \Omega$;
(3) Each Ω_m is a bounded domain with $C^{2,\alpha}$ boundary;
(4) $\operatorname{dist}(0, \partial\Omega \setminus \partial\Omega_m) \to \infty$ as $m \to \infty$.

We can find a number μ, such that the eigenvalue problem

$$\begin{gathered} -\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y) D_{ij}\phi = \lambda(\mu p(\mathbf{x}, y))\phi \quad \text{on } \Omega_m \\ \phi = 0 \quad \text{on } \partial\Omega_m \end{gathered}$$

has a first eigenvalue $\lambda_1(m) < 1$ with eigenfunction ϕ_m. We assume $\max \phi_m = 1$. Let δ be a number such that $0 < \delta \le 1/2$ and $t \le t^\gamma$ for $0 < t < \delta$. Let $u_0 \ge 1$ be a positive solution to (1.8) with $p(\mathbf{x}, y)$ replaced by $\mu p(\mathbf{x}, y)$. Then

$$\begin{gathered} -\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y) D_{ij} w = \mu p(\mathbf{x}, y) w^\gamma \quad \text{on } \Omega_m \\ w = 0 \quad \text{on } \partial\Omega_m \end{gathered} \tag{4.1}$$

has a pair of super and sub solutions $u_0(\mathbf{x}, y)$, $\delta\phi_m$, and $u_0(\mathbf{x}, y) \ge 1 \ge \delta\phi_m$. Thus (4.1) has a positive solution w_m. Since w_m is also a super solution of the equation in (4.1) on Ω_s for all $m > s$, $\delta\phi_s$ is a subsolution of the equation in (4.1) on Ω_s, $\delta\phi_s = 0 \le w_m$ on $\partial\Omega_s$, we have

$$\delta\phi_s \le w_m \quad \text{on } \Omega_s \tag{4.2}$$

for all $m > s$. We also have $w_m \le u_0$ on Ω_s. Therefore a subsequence of w_m will converge to a positive $C^2(\Omega)$ function u that satisfies the equation in (1.1). Furthermore u satisfies (1.7) since u_0 satisfies (1.7). Now we still need to prove that $u \in C^0(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$.

If $(\mathbf{x}_0, y_0) \in \partial\Omega$, we let $O(\mathbf{x}_0, y_0) \in \Pi$ be the $C^{2,\alpha}$ domain chosen in section 2. Then $\partial O(\mathbf{x}_0, y_0) \cap \partial\Omega$ is a neighborhood of $(\mathbf{x}_0, y_0)$ in $\partial\Omega$. Now we choose a C^1 function ψ on $\partial O(\mathbf{x}_0, y_0)$ so that $\psi \ge 0$, $\psi = 0$ in a neighborhood of $(\mathbf{x}_0, y_0) \in \partial O(\mathbf{x}_0, y_0)$ and $\psi \ge u_0$ on $\partial O(\mathbf{x}_0, y_0) \cap \Omega$. Let $w_0 \in C^0(\overline{O(\mathbf{x}_0, y_0)}) \cap C^2(O(\mathbf{x}_0, y_0))$ be the solution of the problem

$$\begin{gathered} -\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y) D_{ij} w_0 = \mu p(\mathbf{x}, y) w_0^\gamma \quad \text{on } O(\mathbf{x}_0, y_0), \\ w_0 = \psi \quad \text{on } \partial O(\mathbf{x}_0, y_0). \end{gathered} \tag{4.3}$$

Then for all m with $O(\mathbf{x}_0, y_0) \subset \Omega_m$, from $w_m \le u_0 \le \psi$ on $\partial O(\mathbf{x}_0, y_0)$, we have

$$w_m \le w_0 \quad \text{on } O(\mathbf{x}_0, y_0).$$

Thus combining this with (4.2), we have that for large fixed s,

$$\delta\phi_s \le u \le w_0 \quad \text{on } O(\mathbf{x}_0, y_0).$$

Since w_0 and $\delta\psi_s$ are in $C^0(\overline{O(\mathbf{x}_0, y_0)})$ and $w_0(\mathbf{x}_0, y_0) = \delta\psi_s(\mathbf{x}_0, y_0) = 0$, u is continuous near $(\mathbf{x}_0, y_0)$ and $u(\mathbf{x}_0, y_0) = 0$. Since $(\mathbf{x}_0, y_0) \in \partial\Omega$ can be arbitrary, we get $u \in C^0(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$.

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