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SOLUTION DEPENDENCE ON PROBLEM PARAMETERS FOR INITIAL-VALUE PROBLEMS ASSOCIATED WITH THE STIELTJES STURM-LIOUVILLE EQUATIONS

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Abstract. We examine properties of solutions to a 2n-dimensional Stieltjes Sturm-Liouville initial-value problem. Existence and uniqueness of a solution has been previously proven, but we present a proof in order to establish properties of boundedness, bounded variation, and continuity. These properties are then used to prove that the solutions depend continuously on the coefficients and on the initial conditions under certain hypotheses. In a future paper, these results will be extended to eigenvalue problems, and we will examine dependence on the endpoints and boundary data in addition to the coefficients. We will find conditions under which the eigenvalues depend continuously and differentiably on these parameters.

1. Introduction

In this work, we examine properties of solutions of generalized 2n-dimensional Sturm-Liouville initial value problems of the form

$$\begin{aligned} dy &= Ay\,dt + dP\,z \\ dz &= (dQ - \lambda dW)y + Dz\,dt \end{aligned} \tag{1.1}$$

on an interval $[a, b]$. Existence and uniqueness of a solution over the class of quasi-continuous functions has already been established [4], but we include part of the proof in section 3 to establish certain bounds and continuity properties of the solution. In section 4, we determine conditions under which the solution depends continuously on the coefficients. This is accomplished by taking a sequence of initial value problems and finding conditions under which the sequence of solutions converges to the solution of the limit problem.

This work generalizes some earlier results. Kong and Zettl [6], [9], and Kong, Wu, and Zettl [7], [8], consider scalar, self-adjoint equations, whereas this work allows for Stieltjes Hamiltonian systems. Thus difference equations are included in our formulation. These authors take sequences of initial value problems to examine dependence of the solution on the problem data, and they require L^1 convergence of the coefficients. For example, in the second order equation $-(py')' + qy = \lambda wy$, the approximate equation $-(p_1y')' + q_1y = \lambda w_1y$ is considered to be close to the original

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equation if $\int_a^b |\frac{1}{p}-\frac{1}{p_1}|+\int_a^b |q-q_1|+\int_a^b |w-w_1|$ is small. We take the same approach of using sequences of initial value problems but allow for more general modes of convergence on the coefficients, with two sequences of coefficients converging weakly in L^1, one sequence converging uniformly, and two sequences converging pointwise. The L^1 convergence used in [6]-[9] is a special case of our convergence conditions. In another related work, Reid [10] addresses this problem for Hamiltonian systems, but we relax his hypotheses on the data and on the modes of convergence. Knotts-Zides [5] extends Reid's results to more general conditions, but her problem is only 2-dimensional and requires $A = D = 0$ in (1.1).

The references mentioned above actually apply the results to eigenvalue problems rather than initial value problems. Likewise, the results we find for initial value problems will be extended to eigenvalue problems in a later paper. We will find conditions under which not only the solutions, but also the eigenvalues, depend continuously on the problem parameters. In addition, conditions will be found under which the eigenvalues depend differentiably on the problem data.

2. Preliminaries

Here we give a preliminary discussion of Stieltjes integrals and some previous results by Hinton [4].

In this work, we take N to be the ring of $2n \times 2n$ matrices and we define the norm as follows: First we define the vector norm by $\|\bar{x}\| := \sum_i |x_i|$ for $\bar{x} = (x_1, \dots, x_{2n})^T$. Then define the norm on N by $\|A\| := \sum_{i,j} |a_{ij}|$ for $A = \{a_{ij}\}$. It follows that for A in N and $\bar{x}$ a $2n$-vector, $\|A\bar{x}\| \le \|A\|\|\bar{x}\|$. Let $\mathbb{R}^m$ be the space of m-vectors.

For some interval $[a, b]$ with $a < b$, a function $F : [a, b] \to N$ is said to be *quasi-continuous* if the left and right limits exist at each interior point of $[a, b]$ and the appropriate one-sided limit exists at each endpoint. A function $F : [a, b] \to N$ is said to be of *bounded variation* if there exists a number K such that

$$\sum_{i=1}^{m} |F(t_i) - F(t_{i-1})| \le K$$

for any partition $a = t_0 < t_1 < \dots < t_m = b$. The greatest lower bound of such constants K is called the *total variation* of F and is denoted $\bigvee_a^b F$ or $\int_a^b \|dF(t)\|$. If a real function F is nondecreasing on an interval $[a, b]$, then F is of bounded variation with $\bigvee_a^b F = F(b)-F(a)$. A function of bounded variation is also quasi-continuous. For $t \ge a$ and a function F of bounded variation on $[a, t]$, define the total variation function $v_F(t) := \bigvee_a^t F$. If F is of bounded variation and is continuous, then v_F is also continuous [11, Theorem 6.26].

If F is a function from $[a, b]$ to N, and if G is a function from $[a, b]$ to N or to $\mathbb{R}^{2n}$, then the *Cauchy-left integral* $(L) \int_a^b dF(x)G(x)$ denotes an element Y of N or $\mathbb{R}^{2n}$ with the following property: for each positive number ϵ, there is a partition $s = \{s_i\}_0^n$ from a to b such that for any partition $t = \{t_i\}_0^m$ that is a refinement of s, then

$$\Big\|Y - \sum_{i=1}^{m} [F(t_i) - F(t_{i-1})]G(t_{i-1})\Big\| < \epsilon$$

The *Cauchy-right integral* $(R) \int_a^b dF(x)G(x)$ is the same except $G(t_{i-1})$ is replaced by $G(t_i)$. The ordinary *Stieltjes integral* $\int_a^b dF(x)G(x)$ denotes an element Y of N

or $\mathbb{R}^{2n}$ with the following property: for each positive number ϵ, there is a partition $s = \{s_i\}_0^n$ from a to b such that for any partition $t = \{t_i\}_0^m$ that is a refinement of s, then

$$\big\|Y - \sum_{i=1}^{m}[F(t_i) - F(t_{i-1})]G(t_i^*)\big\| < \epsilon,$$

for all t_i^* such that $t_i^* \in [t_{i-1}, t_i]$. We can define similarly the Cauchy-left integral $(L)\int_a^b G(x)dF(x)$, the Cauchy-right integral $(R)\int_a^b G(x)dF(x)$, and the Stieltjes integral $\int_a^b G(x)dF(x)$. These integrals are known to be unique if they exist. The Cauchy integrals are known to exist if F is of bounded variation and G is quasi-continuous. If in addition either F or G is continuous, the Stieltjes integral is known to exist. If the Stieltjes integral exists, then the corresponding left-Cauchy and right-Cauchy integrals exist and all three integrals are equal.

Now we state some theorems that will be referenced in this paper [4]. They are stated here without proof.

Theorem 2.1. *If F is of bounded variation from $[a, b]$ to N, G is quasi-continuous from $[a, b]$ to N or to $\mathbb{R}^{2n}$, and f and g are real functions on $[a, b]$ such that for $x \le y$, $\bigvee_x^y F \le f(y) - f(x)$, and $g(x) = \|G(x)\|$, then*

$$\big\|(L)\int_a^b dF(s)G(s)\big\| \le \big|(L)\int_a^b df(s)g(s)\big| \le \big(\bigvee_a^b f\big)\|g\|_{[a,b]},$$

where $\|g\|_{[a,b]} := \sup\{g(x) : x \in [a, b]\}$.

Theorem 2.2. *If $K \ge 0$, h is a real nondecreasing function on $[a, b]$, and if m is a real function on $[a, b]$ bounded above by some $T > 0$ and such that for each x,*

$$m(x) \le K + (L)\int_a^x m(s)\, dh(s),$$

then

$$m(x) \le Ke^{[h(x)-h(a)]}$$

for each x.

Let $\mathcal{F}$ be the set of all functions $F : [a, b] \times [a, b] \to N$ such that

(1) $F(x, x) = I$ for all x,
(2) F is quasi-continuous with respect to its first variable, and
(3) there is a real nondecreasing function g on $[a, b]$ such that $g(a) = 0$ and

$$\|F(t, x) - F(t, y)\| \le \|g(x) - g(y)\|$$

for all t, x, and y. Such a function g is called a *super function* for F.

In this paper, we consider a problem that involves a function $F(t)$, which is not an element of $\mathcal{F}$ since it is a function of only one variable. However, we can define $\tilde{F}(t, x) := I + F(x) - F(t)$, which can be shown to be an element of $\mathcal{F}$. This allows the following two theorems to apply to our case.

Theorem 2.3. *If $F \in \mathcal{F}$, $Q : [a, b] \to N$ is quasi-continuous, $X = L$ or $X = R$, and P is defined on $[a, b]$ by*

$$P(t) = (X)\int_a^t d_sF(t, s)\, Q(s),$$

then P is quasi-continuous. Moreover, if F is continuous with respect to its first variable, then P is continuous.

Theorem 2.4. *Given $F \in \mathcal{F}$, there is a unique $M \in \mathcal{F}$ that is a solution of*

$$M(t,x) = I + (L)\int_x^t d_s F(t,s)\, M(s,x)$$

for all t and x. Moreover, if F is continuous with respect to its first variable, then so is M.

Theorem 2.5. *If $F \in \mathcal{F}$ and G is a quasi-continuous function from $[a,b]$ to N or to $\mathbb{R}^{2n}$, then there is a unique quasi-continuous function Y on $[a,b]$ such that*

$$Y(t) = G(t) + (L)\int_x^t d_s F(t,s)\, Y(s).$$

Now we state two convergence theorems for Stieltjes integrals [3], followed by a convergence theorem for a sequence of functions.

Theorem 2.6 (Helly's Integral Convergence Theorem)**.** *Let f be a continuous function on $[a,b]$ and let $\{g_n\}$ be a sequence of functions, uniformly of bounded variation on $[a,b]$, converging to a function g at every point of $[a,b]$. Then*

$$\lim_{n\to\infty} \int_a^b f\, dg_n = \int_a^b f\, dg.$$

Theorem 2.7 (Osgood's Theorem)**.** *Let g be a function of bounded variation on $[a,b]$ and let $\{f_n\}$ be a sequence of functions which is uniformly bounded and converges pointwise to a function f on $[a,b]$. If $\int_a^b f_n\, dg$ and $\int_a^b f\, dg$ exist, then*

$$\lim_{n\to\infty} \int_a^b f_n\, dg = \int_a^b f\, dg.$$

Theorem 2.8 (Helly's Pointwise Convergence Theorem)**.** *If f_n is a sequence of functions, uniformly of bounded variation on $[a,b]$ such that $f_n(a)$ is bounded in n, then there exists a subsequence f_{n_m} and a function f of bounded variation such that $\lim_{m\to\infty} f_{n_m} = f$ at every point of $[a,b]$.*

Here we introduce some notation that will be used throughout this paper. Let $\mathcal{P}[a,b]$ denote the set of all partitions of the interval $[a,b]$. Also, define, for f integrable, $I_f(t) := \int_a^t f$.

3. The Initial Value Problem

We consider the system of $2n$ equations

$$\begin{aligned} dy &= Ay\, dt + dP\, z \\ dz &= (dQ - \lambda dW)y + Dz\, dt \end{aligned} \tag{3.1}$$

which can be written as the Stieltjes integral equation

$$\begin{bmatrix} y(t,\lambda) \\ z(t,\lambda) \end{bmatrix} = \begin{bmatrix} y(a,\lambda) \\ z(a,\lambda) \end{bmatrix} + \int_a^t \begin{bmatrix} A(s)ds & dP(s) \\ dM(s,\lambda) & D(s)ds \end{bmatrix} \begin{bmatrix} y(s,\lambda) \\ z(s,\lambda) \end{bmatrix} \tag{3.2}$$

on $[a,b]\times K$, where K is a compact set in $\mathbb{C}$, and $M(t,\lambda)=Q(t)-\lambda W(t)$. Here, y and z are n-vectors and A, P, Q, W, and D are $n\times n$ real matrices. We require the following conditions on the coefficients:

$$\begin{aligned}
&A, D \in L_1([a,b]);\\
&P = P^T \text{ is continuous and nondecreasing with } P(a)=0;\\
&Q = Q^T \text{ is of bounded variation on } [a,b];\\
&W = W^T \text{ is nondecreasing with } W(a)=0,
\end{aligned} \tag{3.3}$$

Remark 3.1. (1) It is a consequence of a matrix function being of bounded variation that each component is of bounded variation.

(2) When we say a real symmetric matrix A is positive, we mean that all of the eigenvalues of A are positive. This is equivalent to the condition that $\langle A\bar{x},\bar{x}\rangle>0$ for all $\bar{x}\neq\bar{0}$. The meaning of a matrix function $A(t)$ being nondecreasing follows accordingly, i.e., $A(t)$ is nondecreasing if $A(t_2)-A(t_1)$ is nonnegative for $t_2\geq t_1$.

(3) The symmetry condition on Q, P, and W is needed later for self-adjoint problems, but it is not needed to prove existence of a solution.

In Reid's paper [10], the derivatives of P, Q, and W are required to be L_1 functions. We are allowing for more generality in the coefficients that does not require a Banach space.

We define the following terms:

$$L := \Big\| \begin{bmatrix} y(a,\lambda)\\ z(a,\lambda)\end{bmatrix} \Big\| \quad F(t,\lambda) := \begin{bmatrix} \int_a^t A(s)\,ds & P(t)\\ M(t,\lambda) & \int_a^t D(s)\,ds \end{bmatrix}$$

$$f(t,\lambda) := \int_a^t \|A(s)\|\,ds + \int_a^t \|D(s)\|\,ds + \bigvee_a^t P + \bigvee_a^t M(\cdot,\lambda).$$

Note that $\bigvee_a^t M(\cdot,\lambda)\leq \bigvee_a^t Q + |\lambda|\bigvee_a^t W$.

Suppose we fix λ and let $Y(t) := \begin{bmatrix} y(t,\lambda)\\ z(t,\lambda)\end{bmatrix}$. Then (3.2) can be written in the form $Y(t)=Y(a)+\int_a^t dF(s)\,Y(s)$. We now show that for fixed λ, $\tilde{F}(t,x) := I+F(x,\lambda)-F(t,\lambda)$ is an element of the set $\mathcal{F}$ which was defined in section 2. The first condition, $\tilde{F}(x,x)=I$ is clearly satisfied. For the second condition, it suffices to show that $F(t,\lambda)$ is quasi-continuous in its first variable. We know P is continuous by assumption. The remaining elements of F are now shown to be of bounded variation, which implies they are quasi-continuous. We know $\int_a^t A$ and $\int_a^t D$ are of bounded variation since A, $D\in L_1$, and Q is of bounded variation by assumption. P and W are of bounded variation because they are nondecreasing. The third condition is satisfied for $g(t):=f(t,x)$. Since $\tilde{F}(t,x)\in\mathcal{F}$, Theorems 2.3, 2.4, and 2.5 apply to our problem, meaning that equation 3.1 has a unique quasi-continuous solution. We repeat a proof of existence and uniqueness to establish certain uniform bounds, to determine the continuity properties of the solution, and to establish a Lipschitz condition with respect to a spectral parameter.

Define successive approximations for $k = 1, 2, 3, \dots$ as follows: Given initial conditions $y(a,\lambda)$ and $z(a,\lambda)$, let

$$\begin{bmatrix} y^{(0)}(t,\lambda) \\ z^{(0)}(t,\lambda) \end{bmatrix} = \begin{bmatrix} y(a,\lambda) \\ z(a,\lambda) \end{bmatrix} \tag{3.4}$$

$$\begin{bmatrix} y^{(k)}(t,\lambda) \\ z^{(k)}(t,\lambda) \end{bmatrix} = \begin{bmatrix} y^{(0)}(t,\lambda) \\ z^{(0)}(t,\lambda) \end{bmatrix} + \int_a^t \begin{bmatrix} A(s)\,ds & dP(s) \\ dM(s,\lambda) & D(s)\,ds \end{bmatrix} \begin{bmatrix} y^{(k-1)}(s,\lambda) \\ z^{(k-1)}(s,\lambda) \end{bmatrix} \tag{3.5}$$

First we examine some properties of the successive approximations. The proof of the following lemma follows by a standard argument.

Lemma 3.2. *If successive approximations are defined as in* (3.4), (3.5) *subject to the conditions in* (3.3)*, then*

$$\Big\| \begin{bmatrix} y^{(k)}(t,\lambda) \\ z^{(k)}(t,\lambda) \end{bmatrix} - \begin{bmatrix} y^{(k-1)}(t,\lambda) \\ z^{(k-1)}(t,\lambda) \end{bmatrix} \Big\| \le L \frac{f(t,\lambda)^k}{k!}$$

for $k = 1, 2, 3, \dots$

Lemma 3.3. *If successive approximations are defined as in* (3.4), (3.5) *subject to the conditions in* (3.3)*, then* $\Big\| \begin{bmatrix} y^{(k)}(t,\lambda) \\ z^{(k)}(t,\lambda) \end{bmatrix} \Big\|$ *and* $\bigvee_a^t \begin{bmatrix} y^{(k)}(\cdot,\lambda) \\ z^{(k)}(\cdot,\lambda) \end{bmatrix}$ *are bounded independently of* k *and* $(t,\lambda) \in [a,b] \times K$.

Proof. Using Lemma 3.2,

$$\begin{aligned} \Big\| \begin{bmatrix} y^{(k)}(t,\lambda) \\ z^{(k)}(t,\lambda) \end{bmatrix} \Big\| &\le \Big\| \begin{bmatrix} y^{(0)}(t,\lambda) \\ z^{(0)}(t,\lambda) \end{bmatrix} \Big\| + \sum_{i=1}^{k} \Big\| \begin{bmatrix} y^{(i)}(t,\lambda) \\ z^{(i)}(t,\lambda) \end{bmatrix} - \begin{bmatrix} y^{(i-1)}(t,\lambda) \\ z^{(i-1)}(t,\lambda) \end{bmatrix} \Big\| \\ &\le L + L \sum_{i=1}^{k} \frac{f(t,\lambda)^i}{i!} \le L e^{f(t,\lambda)}, \end{aligned}$$

Recall also that

$$f(t,\lambda) \le \int_a^t \|A\| + \int_a^b \|D\| + \bigvee_a^b P + \bigvee_a^b Q + |\lambda| \bigvee_a^b W, \tag{3.6}$$

which is bounded independently of $(t,\lambda) \in [a,b] \times K$.

We now turn to the bound on the total variaton. Let $T = \{t_i\}_0^m \in \mathcal{P}[a,t]$. Using Theorem 2.1 and letting C be a bound on $\Big\| \begin{bmatrix} y^{(k)}(\cdot,\lambda) \\ z^{(k)}(\cdot,\lambda) \end{bmatrix} \Big\|$,

$$\begin{aligned} \Big\| \bigvee_a^t \begin{bmatrix} y^{(k)}(\cdot,\lambda) \\ z^{(k)}(\cdot,\lambda) \end{bmatrix} \Big\| &= \sup_{T \in P[a,t]} \sum_{i=1}^{m} \Big\| \begin{bmatrix} y^{(k)}(t_i) \\ z^{(k)}(t_i) \end{bmatrix} - \begin{bmatrix} y^{(k)}(t_{i-1}) \\ z^{(k)}(t_{i-1}) \end{bmatrix} \Big\| \\ &= \sup_{T \in P[a,t]} \sum_{i=1}^{m} \Big\| \int_{t_{i-1}}^{t_i} \begin{bmatrix} A(s)\,ds & dP(s) \\ dM(s,\lambda) & D(s)\,ds \end{bmatrix} \begin{bmatrix} y^{(k-1)}(s,\lambda) \\ z^{(k-1)}(s,\lambda) \end{bmatrix} \Big\| \\ &\le \sup_{T \in P[a,t]} C \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} dv_F(s) \\ &= C \int_a^t dv_F(s) \le C \int_a^t df(s,\lambda) = C f(t,\lambda), \end{aligned}$$

since Δv_F is bounded by Δf. Independence follows from (3.6). $\square$

Now we examine the continuity of solutions, starting by proving that each successive approximation is quasi-continuous in the first variable. This will be used in the subsequent theorem to prove the solution is quasi-continuous.

Lemma 3.4. *As a function of the first variable, $y^{(k)}(t,\lambda)$ is continuous and $z^{(k)}(t,\lambda)$ is quasi-continuous on $[a,b]$, for $k=0,1,2,\dots$. If Q and W are continuous, then $z^{(k)}(t,\lambda)$ is continuous on $[a,b]$, for $k=0,1,2,\dots$.*

Proof. The quasi-continuity of $y^{(k)}(\cdot,\lambda)$ and $z^{(k)}(\cdot,\lambda)$ follows from the fact that they are of bounded variation (Lemma 3.3). Now we prove that $y^{(k)}(\cdot,\lambda)$ is actually continuous.

Fix $\lambda\in K$. By definition of successive approximations, for $k=1,2,3,\dots$

$$y^{(k)}(t,\lambda)=y(a,\lambda)+\int_a^t A(s)\,y^{(k-1)}(s,\lambda)\,ds+\int_a^t dP(s)\,z^{(k-1)}(s,\lambda).$$

Let $g(t)=\int_a^t A(s)\,y^{(k-1)}(s,\lambda)\,ds$, and $h(t)=\int_a^t dP(s)\,z^{(k-1)}(s,\lambda)$. To show that g is continuous, note that

$$\|g(t_2)-g(t_1)\|=\Big\|\int_{t_1}^{t_2} A(s)\,y^{(k-1)}(s,\lambda)\,ds\Big\|\le C\int_{t_1}^{t_2}\|A(s)\|\,ds,$$

where C is a uniform bound on $\Big\|\begin{bmatrix}y^{(k)}(t,\lambda)\\ z^{(k)}(t,\lambda)\end{bmatrix}\Big\|$. Since $A\in L_1([a,b])$, $I_{\|A\|}(t):=\int_a^t\|A(s)\|\,ds$ is absolutely continuous, which implies g is continuous.

Now for h, we have

$$\begin{aligned}\|h(t_2)-h(t_1)\|&=\Big\|\int_{t_1}^{t_2} dP(s)\,z^{(k-1)}(s,\lambda)\Big\|\\&\le\int_{t_1}^{t_2} dv_P(s)\,\|z^{(k-1)}(s,\lambda)\|\\&\le C(v_P(t_2)-v_P(t_1)).\end{aligned}$$

Since P is continuous, v_P is also continuous; hence h is continuous. Now both g and h are continuous, so $y^{(k)}(t,\lambda)$ is continuous as a function of t.

If Q and W are continuous, then $z^{(k)}(t,\lambda)$ can be shown to be continuous using a similar argument. □

Theorem 3.5. *The initial value problem* (3.2) *has a unique solution in the space of quasi-continuous functions. This solution is bounded in norm and in total variation independently of t and λ on $[a,b]\times K$.*

Proof. We prove the existence of a solution by showing it is the limit of the sequence of successive approximations. By Lemma 3.2, we have for $p>k$,

$$\begin{aligned}\Big\|\begin{bmatrix}y^{(p)}(t,\lambda)\\ z^{(p)}(t,\lambda)\end{bmatrix}-\begin{bmatrix}y^{(k)}(t,\lambda)\\ z^{(k)}(t,\lambda)\end{bmatrix}\Big\|&\le\sum_{i=k+1}^{p}\Big\|\begin{bmatrix}y^{(i)}(t,\lambda)\\ z^{(i)}(t,\lambda)\end{bmatrix}-\begin{bmatrix}y^{(i-1)}(t,\lambda)\\ z^{(i-1)}(t,\lambda)\end{bmatrix}\Big\|\\&\le L\sum_{i=k+1}^{p}\frac{f(t,\lambda)^i}{i!}\le L\frac{f(t,\lambda)^{k+1}}{(k+1)!}e^{f(t,\lambda)}\\&\le L\frac{f(b,\lambda)^{k+1}}{(k+1)!}e^{f(b,\lambda)}\to 0\quad\text{as } k\to\infty.\end{aligned}$$

Thus $\{[y^{(k)}, z^{(k)}]\}$ is uniformly Cauchy in $[a,b]$. Taking the limit as $k \to \infty$ in equation (3.5) and using the fact that the convergence of the successive approximations is uniform, we get that

$$\begin{bmatrix} y(t,\lambda) \\ z(t,\lambda) \end{bmatrix} := \lim_{k\to\infty} \begin{bmatrix} y^{(k)}(t,\lambda) \\ z^{(k)}(t,\lambda) \end{bmatrix}$$

is a solution. We know that each successive approximation is quasi-continuous in t and uniformly bounded in norm and in total variation by Lemma 3.3, so this solution has these same properties.

The proof of uniqueness is a standard argument, using the Gronwall inequality from Theorem 2.2. □

The first part of the following corollary follows from the fact that the functions $y^{(k)}(t,\lambda)$ are continuous with respect to t and that they converge uniformly to $y(t,\lambda)$ as $k \to \infty$. The second part follows from Lemma 3.4 and the uniform convergence of the successive approximations.

Corollary 3.6. (1) *$y(t,\lambda)$ is continuous with respect to t.*
(2) *If Q and W are continuous on $[a,b]$, then $z(t,\lambda)$ is continuous for $t \in [a,b]$.*

We conclude with some additional properties of solutions. First a Lipschitz condition in λ, and then a property concerning total variation of the solution.

Lemma 3.7. *For any λ_1 and λ_2 in a compact set $K \subseteq \mathbb{C}$, we have the Lipschitz condition*

$$\Big\| \begin{bmatrix} y(t,\lambda_2) \\ z(t,\lambda_2) \end{bmatrix} - \begin{bmatrix} y(t,\lambda_1) \\ z(t,\lambda_1) \end{bmatrix} \Big\| \le c|\lambda_2 - \lambda_1|,$$

where c is a constant independent of $t \in [a,b]$ and $\lambda_1, \lambda_2 \in K$.

Proof. Let

$$\begin{aligned} \phi(t) &:= \Big\| \begin{bmatrix} y(t,\lambda_2) \\ z(t,\lambda_2) \end{bmatrix} - \begin{bmatrix} y(t,\lambda_1) \\ z(t,\lambda_1) \end{bmatrix} \Big\| \\ &= \|y(t,\lambda_2) - y(t,\lambda_1)\| + \|z(t,\lambda_2) - z(t,\lambda_1)\|. \end{aligned}$$

Using the equation $y(t,\lambda_i) = y(a,\lambda_i) + \int_a^t A(s)\,y(s,\lambda_i)\,ds + \int_a^t dP(s)\,z(s,\lambda_i)$ for $i = 1,2$ and the fact that $y(a,\lambda_1) = y(a,\lambda_2)$, we have

$$y(t,\lambda_2) - y(t,\lambda_1) = \int_a^t A(s)\,[y(s,\lambda_2) - y(s,\lambda_1)]\,ds + \int_a^t dP(s)\,[z(s,\lambda_2) - z(s,\lambda_1)].$$

Similarly, we have

$$\begin{aligned} &z(t,\lambda_2) - z(t,\lambda_1) \\ &= \int_a^t dQ(s)\,[y(s,\lambda_2) - y(s,\lambda_1)] \\ &\quad - \int_a^t dW(s)\,[\lambda_2 y(s,\lambda_2) - \lambda_1\,y(s,\lambda_1)] + \int_a^t D(s)\,[z(s,\lambda_2) - z(s,\lambda_1)]\,ds. \end{aligned}$$

By adding and subtracting the term $\int_a^t dW(s)\,\lambda_1\, y(s,\lambda_2)$, we get

$$\begin{aligned}
&z(t,\lambda_2)-z(t,\lambda_1)\\
&=\int_a^t dQ(s)\,[y(s,\lambda_2)-y(s,\lambda_1)]-\int_a^t dW(s)\,(\lambda_2-\lambda_1)\,y(s,\lambda_2)\\
&-\int_a^t dW(s)\lambda_1\,[y(s,\lambda_2)-y(s,\lambda_1)]+\int_a^t D(s)\,[z(s,\lambda_2)-z(s,\lambda_1)]\,ds.
\end{aligned}$$

Let $C>0$ be a bound on $\|\begin{bmatrix} y(t,\lambda)\\ z(t,\lambda)\end{bmatrix}\|$, from Theorem 3.5. Then

$$\begin{aligned}
\phi(t)&\le \int_a^t \|A(s)\|\,\|y(s,\lambda_2)-y(s,\lambda_1)\|\,ds+\int_a^t dv_P(s)\,\|z(s,\lambda_2)-z(s,\lambda_1)\|\\
&\quad+\int_a^t dv_Q(s)\,\|y(s,\lambda_2)-y(s,\lambda_1)\|+|\lambda_2-\lambda_1|\int_a^t dv_W(s)\,\|y(s,\lambda_2)\|\\
&\quad+|\lambda_1|\int_a^t dv_W(s)\,\|y(s,\lambda_2)-y(s,\lambda_1)\|+\int_a^t \|D(s)\|\,\|z(s,\lambda_2)-z(s,\lambda_1)\|\,ds\\
&\le \int_a^t dv_A(s)\,\phi(s)+\int_a^t dv_P(s)\,\phi(s)+(L)\int_a^t dv_Q(s)\,\phi(s)+C\,|\lambda_2-\lambda_1|\bigvee_a^b W\\
&\quad+|\lambda_1|\,(L)\int_a^t dv_W(s)\,\phi(s)+\int_a^t dv_D(s)\,\phi(s).
\end{aligned}$$

By Theorem 2.2, it follows that

$$\phi(t)\le \big(C\,|\lambda_2-\lambda_1|\bigvee_a^b W\big)\,e^{m(t)},$$

where $m(t)=v_A(t)+v_P(t)+v_Q(t)+kv_W(t)+v_D(t)$, where $k=\sup_{\lambda\in K}\lambda$. Then $\phi(t)\le c\,|\lambda_2-\lambda_1|$, where $c=C\bigvee_a^b W\,e^{m(b)}$. □

Theorem 3.8. *There is a nondecreasing function q on $[a,b]$ such that $\bigvee_{x_1}^{x_2} y(x,\lambda)\le q(x_2)-q(x_1)$ and $\bigvee_{x_1}^{x_2} z(x,\lambda)\le q(x_2)-q(x_1)$ for $x_1<x_2$ and all $\lambda\in K$.*

Proof. Fix $\lambda\in K$. For any $T=\{t_i\}_0^m\in\mathcal{P}[x_1,x_2]$,

$$\begin{aligned}
\bigvee_{x_1}^{x_2} y(\cdot,\lambda)&=\sup_{T\in\mathcal{P}[x_1,x_2]}\sum_{i=1}^m \|y(t_i,\lambda)-y(t_{i-1},\lambda)\|\\
&=\sup_{T\in\mathcal{P}[x_1,x_2]}\sum_{i=1}^m \Big\|\int_{t_{i-1}}^{t_i} A(s)\,y(s,\lambda)\,ds+\int_{t_{i-1}}^{t_i} dP(s)\,z(s,\lambda)\Big\|\\
&\le \sup_{T\in\mathcal{P}[x_1,x_2]}\sum_{i=1}^m \int_{t_{i-1}}^{t_i} \|A(s)\,y(s,\lambda)\,ds\|+\int_{t_{i-1}}^{t_i} \|dP(s)\,z(s,\lambda)\|\\
&=\int_{x_1}^{x_2} \|A(s)\,y(s,\lambda)\,ds\|+\int_{x_1}^{x_2} \|dP(s)\,z(s,\lambda)\|\\
&\le C\int_{x_1}^{x_2} \|A(s)\|\,ds+C\bigvee_{x_1}^{x_2} P\\
&=q_1(x_2)-q_1(x_1),
\end{aligned}$$

where C is a bound on $\|\begin{bmatrix} y(t,\lambda) \\ z(t,\lambda) \end{bmatrix}\|$ and $q_1(x) = C\int_a^x \|A(s)\|\,ds + C\bigvee_a^x P$. Note that q_1 is continuous since P is continuous [11, Theorem 6.26] and since $A \in L_1([a,b])$. Now

$$
\begin{aligned}
&\bigvee_{x_1}^{x_2} z(x,\lambda) \\
&= \sup_{T\in\mathcal{P}[x_1,x_2]} \sum_{i=1}^{m} \|z(t_i,\lambda) - z(t_{i-1},\lambda)\| \\
&= \sup_{T\in\mathcal{P}[x_1,x_2]} \sum_{i=1}^{m} \Big\| \int_{t_{i-1}}^{t_i} (dQ(s) - \lambda\, dW(s))\, y(s,\lambda) + \int_{t_{i-1}}^{t_i} D(s)\, z(s,\lambda)\, ds \Big\| \\
&\le \sup_{T\in\mathcal{P}[x_1,x_2]} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \|(dQ(s) - \lambda\, dW(s))\, y(s,\lambda)\| + \int_{t_{i-1}}^{t_i} \|D(s)\, z(s,\lambda)\, ds\| \\
&\le C\int_{x_1}^{x_2} \|dQ(s)\| + C\,(\max_{\lambda\in K}|\lambda|)\int_{x_1}^{x_2} \|dW(s)\| + C\int_{x_1}^{x_2} \|D(s)\|\, ds \\
&= q_2(x_2) - q_2(x_1),
\end{aligned}
$$

where $q_2(x) = C\bigvee_a^x Q + C\,(\max_{\lambda\in K}\|\lambda\|)\bigvee_a^x W + C\int_a^x \|D(s)\|\,ds$. Note that q_2 is nondecreasing but may have discontinuities due to Q and W. Then $q(x) := q_1(x) + q_2(x)$ satisfies the conclusion of the theorem. □

4. A Sequence of Initial Value Problems

In section 3, we examined the continuity of a solution to the initial value problem as a function of t and of λ. Now we study continuity as a function of the problem coefficients. To accomplish this, we define a sequence of initial value problems satisfying the same conditions as the problem in the previous section (therefore the results from section 3 apply to each problem in the sequence). We will show that the sequence of solutions converges to the solution of the limit problem.

The sequence of problems is defined as follows, for $n = 1, 2, 3, \dots$:

$$
\begin{aligned}
dy_n &= A_n\, y_n\, dt + dP_n\, z_n \\
dz_n &= (dQ_n - \lambda\, dW_n)\, y_n + D_n\, z_n\, dt
\end{aligned}
\tag{4.1}
$$

on $[a,b]\times K$, which written as a Stieltjes integral equation has the form

$$
\begin{bmatrix} y_n(t,\lambda) \\ z_n(t,\lambda) \end{bmatrix} = \begin{bmatrix} y_n(a,\lambda) \\ z_n(a,\lambda) \end{bmatrix} + \int_a^t \begin{bmatrix} A_n(s)\, ds & dP_n(s) \\ dM_n(s,\lambda) & D_n(s)\, ds \end{bmatrix} \begin{bmatrix} y_n(s,\lambda) \\ z_n(s,\lambda) \end{bmatrix}
$$

where

$$\begin{gathered}
A_n \to A \quad \text{weakly in } L_1([a,b]); \\
D_n \to D \quad \text{weakly in } L_1([a,b]); \\
P_n \to P \quad \text{uniformly in } [a,b]; \\
Q_n \to Q \quad \text{pointwise in } [a,b]; \\
W_n \to W \quad \text{pointwise in } [a,b]; \\
y_n(a,\lambda) \to y(a,\lambda); \quad z_n(a,\lambda) \to z(a,\lambda); \\
\bigvee_a^b Q_n \le \tilde{Q}, \quad \bigvee_a^b W_n \le \tilde{W} \quad \text{for all } n,
\end{gathered} \tag{4.2}$$

for some constants $\tilde{Q}$ and $\tilde{W}$. Note that A_n, P_n, $M_n = Q_n - \lambda W_n$, and D_n satisfy the same conditions as A, P, M, and D, respectively, as stated in (3.3). Also note that we require Q_n and W_n to be uniformly of bounded variation. The modes of convergence required here are more inclusive than in previous work [10], which is appropriate due to the increased generality in the classes of coefficients.

For the remainder of this section we fix λ and write $\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} y(t,\lambda) \\ z(t,\lambda) \end{bmatrix}$, understanding that there is still a dependence on λ. We define the following terms for use in the proof:

$$\begin{gathered}
L = \Big\| \begin{bmatrix} y(a) \\ z(a) \end{bmatrix} \Big\|; \quad L_n = \Big\| \begin{bmatrix} y_n(a) \\ z_n(a) \end{bmatrix} \Big\|; \\
f(t) = \int_a^t \|A(s)\|\, ds + \int_a^t \|D(s)\|\, ds + \bigvee_a^t P + \bigvee_a^t M; \\
f_n(t) = \int_a^t \|A_n(s)\|\, ds + \int_a^t \|D_n(s)\|\, ds + \bigvee_a^t P_n + \bigvee_a^t M_n.
\end{gathered}$$

Recall that in Lemma 3.3 we proved the existence of a uniform bound on all successive approximations for one initial value problem. Now we extend that uniform bound to all successive approximations for all of the initial value problems in the sequence.

Lemma 4.1. *Let* $\begin{bmatrix} y_n^{(k)}(t) \\ z_n^{(k)}(t) \end{bmatrix}$, $k = 1, 2, 3, \dots$ *be the kth successive approximation for the nth problem in the sequence* (4.1) *for* $n = 1, 2, 3, \dots$, *defined in the same manner as for problem* (3.1). *Then under assumptions* (4.2), $\Big\| \begin{bmatrix} y_n^{(k)}(t) \\ z_n^{(k)}(t) \end{bmatrix} \Big\|$ *and* $\bigvee_a^b \begin{bmatrix} y_n^{(k)} \\ z_n^{(k)} \end{bmatrix}$ *are bounded independently of* n, k, *and* t.

Proof. By the proof of Lemma 3.3, we have

$$\Big\| \begin{bmatrix} y_n^{(k)}(t) \\ z_n^{(k)}(t) \end{bmatrix} \Big\| \le L_n e^{f_n(b)}; \quad \bigvee_a^t \begin{bmatrix} y_n^{(k)} \\ z_n^{(k)} \end{bmatrix} \le L_n e^{f_n(b)} f_n(b).$$

Therefore, the proof will be complete when we find a uniform bound on $\{L_n\}_1^\infty$ and $\{f_n(b)\}_1^\infty$. The sequence $\{L_n\}_1^\infty$ is bounded because of the assumption that $L_n \to L$.

For $f_n(b) = \int_a^b \|A_n(s)\|\,ds + \int_a^b \|D_n(s)\|\,ds + \bigvee_a^b P_n + \bigvee_a^b M_n$, first observe that the weak convergence of $\{A_n\}_1^\infty$ and $\{D_n\}_1^\infty$ guarantees a uniform bound on $\int_a^b \|A_n(s)\| ds$ and $\int_a^b \|D_n(s)\| ds$ for all n [10, p.430]. If $P(t) = \{P_{ij}\}$, then for $T = \{t_i\}_1^m \in \mathcal{P}[a,b]$,

$$\begin{aligned}\bigvee_a^b P &= \sup_{T\in\mathcal{P}[a,b]} \sum_{k=1}^m \|P(t_k) - P(t_{k-1})\| \\ &= \sup_{T\in\mathcal{P}[a,b]} \sum_{i,j=1}^n \sum_{k=1}^m |P_{ij}(t_k) - P_{ij}(t_{k-1})|.\end{aligned}$$

Since P is symmetric and nondecreasing, together with the fact that $P(t) \geq P(a) = 0$, any component of P is bounded by the square root of the product of the corresponding diagonal entries (which are nonnegative), i.e., $|P_{ij}| \leq \sqrt{P_{ii}P_{jj}}$. Then using the fact that $\sqrt{ab} \leq \frac{1}{2}(a+b)$ for any positive numbers a and b, we have $|P_{ij}| \leq \frac{1}{2}(P_{ii} + P_{jj})$. Then continuing from above, we have

$$\begin{aligned}\bigvee_a^b P &\leq \frac{1}{2} \sum_{i,j=1}^n \sup_{T\in\mathcal{P}[a,b]} \sum_{k=1}^m |P_{ii}(t_k) - P_{ii}(t_{k-1})| + |P_{jj}(t_k) - P_{jj}(t_{k-1})| \\ &= \frac{1}{2} \sum_{i,j=1}^n (\bigvee_a^b P_{ii} + \bigvee_a^b P_{jj}).\end{aligned}$$

Now define $\hat{P}(t) := \sum_{i=1}^n P_{ii}(t)$. Since the diagonal entries of P are nondecreasing and have initial value zero, we have $\sum_{i=1}^n \bigvee_a^b P_{ii} = \bigvee_a^b \hat{P}$. Therefore,

$$\bigvee_a^b P \leq \frac{1}{2} \sum_{i,j=1}^n \Big(\bigvee_a^b P_{ii} + \bigvee_a^b P_{jj}\Big) = \frac{1}{2}\Big(n\bigvee_a^b \hat{P} + n\bigvee_a^b \hat{P}\Big) = n\bigvee_a^b \hat{P} = n\hat{P}(b),$$

with the last equality holding because $\hat{P}$ is continuous and nondecreasing with $\hat{P}(a) = 0$. Similarly,

$$\bigvee_a^b P_j \leq n\hat{P}_j(b).$$

Since $\hat{P}_j(b) \to \hat{P}(b)$, there exists a bound on $\{\hat{P}_j(b)\}_1^\infty$. Hence there exists a bound on $\bigvee_a^b P_j$ for all j.

Recall we have a uniform bound on $\bigvee_a^b M_n$ for all n by assumption. Therefore, $\{f_n(b)\}_{n=1}^\infty$ is bounded and the proof is completed. $\square$

In preparation for showing the convergence of the sequence of solutions, we now show the same is true for each set of successive approximations.

Lemma 4.2. *Under the assumptions in* (4.2)*, for each* $k = 0, 1, 2, \dots$.

(1) $y_n^{(k)}(t) \to y^{(k)}(t)$ *uniformly on* $[a,b]$ *as* $n \to \infty$*, and*
(2) $z_n^{(k)}(t) \to z^{(k)}(t)$ *pointwise on* $[a,b]$ *as* $n \to \infty$.

Proof. The proof is by induction on k. First, note that

$$\|y^{(0)}(t) - y_n^{(0)}(t)\| = \|y(a) - y_n(a)\| \to 0,$$
$$\|z^{(0)}(t) - z_n^{(0)}(t)\| = \|z(a) - z_n(a)\| \to 0$$

by assumption. Assume the lemma to be true for some k, and now consider y and z separately to show the lemma is true for $k+1$.
(1) By definition of successive approximations,

$$\begin{aligned} y^{(k+1)}(t) - y_n^{(k+1)}(t) = y(a) &+ \int_a^t [A(s)\, y^{(k)}(s)\, ds + dP(s)\, z^{(k)}(s)] \\ &- y_n(a) - \int_a^t [A_n(s)\, y_n^{(k)}(s)\, ds + dP_n(s)\, z_n^{(k)}(s)]. \end{aligned}$$

Then

$$\begin{aligned} &y^{(k+1)}(t) - y_n^{(k+1)}(t) \\ &= [y(a) - y_n(a)] + \int_a^t [dP(s) - dP_n(s)]\, z^{(k)}(s) \\ &\quad - \int_a^t dP_n(s)\, [z_n^{(k)}(s) - z^{(k)}(s)] + \int_a^t [A(s) - A_n(s)]\, y^{(k)}(s)\, ds \\ &\quad - \int_a^t A_n(s)\, [y_n^{(k)}(s) - y^{(k)}(s)]\, ds. \end{aligned} \tag{4.3}$$

We will show that each of these terms converges to zero uniformly in $[a, b]$. First, $\|y(a) - y_n(a)\| \to 0$ by assumption.

For the second term in (4.3), we use integration by parts and the fact that $P(a) = P_n(a) = 0$ to write

$$\begin{aligned} &\int_a^t [dP(s) - dP_n(s)]\, z^{(k)}(s) \\ &= -\int_a^t [P(s) - P_n(s)]\, dz^{(k)}(s) + [P(t) - P_n(t)]\, z^{(k)}(t). \end{aligned} \tag{4.4}$$

Since P and P_n are continuous and therefore bounded on $[a, b]$,

$$\begin{aligned} &\Big\| \int_a^t [dP(s) - dP_n(s)]\, z^{(k)}(s) \Big\| \\ &\le \Big(\max_{t \in [a,b]} \|P(t) - P_n(t)\| \Big) \bigvee_a^b z^{(k)} + \|P(t) - P_n(t)\|\, \|z^{(k)}(t)\|. \end{aligned}$$

This converges to zero as $n \to \infty$ since $\bigvee_a^b z^{(k)}$ and $\|z^{(k)}(t)\|$ are bounded (by Lemma 3.3) and since $P_n \to P$ uniformly by assumption.

Rewrite the third term in (4.3) as

$$\begin{aligned} &\int_a^t dP_n(s)\, [z_n^{(k)}(s) - z^{(k)}(s)] \\ &= \int_a^t [dP_n(s) - dP(s)]\, [z_n^{(k)}(s) - z^{(k)}(s)] + \int_a^t dP(s)\, [z_n^{(k)}(s) - z^{(k)}(s)]. \end{aligned} \tag{4.5}$$

Now treat the first of the two integrals on the right-hand side of (4.5) the same way as in (4.4). Performing integration by parts and taking the norm, we get

$$\begin{aligned}
&\Big\| \int_a^t [dP_n(s) - dP(s)]\,[z_n^{(k)}(s) - z^{(k)}(s)] \Big\| \\
&\le \Big(\max_{t\in[a,b]} \|P_n(t) - P(t)\| \Big) \bigvee_a^b [z_n^{(k)} - z^{(k)}] + \|P_n(t) - P(t)\|\, \|z_n^{(k)}(t) - z^{(k)}(t)\|,
\end{aligned}$$

which converges uniformly to zero for the same reasons as in (4.4).

The pointwise convergence of the second integral on the right-hand side of (4.5) follows from Theorem 2.7. This theorem applies since P is of bounded variation, since $[z_n^{(k)}(s) - z^{(k)}(s)]$ is uniformly bounded, and since $z_n^{(k)}(t) \to z^{(k)}(t)$ pointwise by induction hypothesis. Now we use the Ascoli-Arzela Theorem to show the convergence is actually uniform and not only pointwise. The sequence $\{\int_a^t dP(s)\,[z_n^{(k)}(s) - z^{(k)}(s)]\}_{n=1}^{\infty}$ is uniformly bounded on $[a,b]$ since

$$\Big\| \int_a^t dP(s)\,[z_n^{(k)}(s) - z^{(k)}(s)] \Big\| \le \Big(\sup_{t\in[a,b]} \|z_n^{(k)}(t) - z^{(k)}(t)\| \Big) \bigvee_a^b P.$$

This term is finite since $z^{(k)}$ and $z_n^{(k)}$ are quasi-continuous by Lemma 3.4 and are uniformly bounded by Lemma 4.1. To show the sequence is equicontinuous, fix n and k and let $g(t) = \int_a^t dP(s)\,[z_n^{(k)}(s) - z^{(k)}(s)]$. Now

$$\begin{aligned}
\|g(t_2) - g(t_1)\| &= \Big\| \int_{t_1}^{t_2} dP(s)\,[z_n^{(k)}(s) - z^{(k)}(s)] \Big\| \\
&\le \Big(\sup_{t\in[a,b]} \|z_n^{(k)}(t) - z^{(k)}(t)\| \Big) \bigvee_{t_1}^{t_2} P \\
&= \Big(\sup_{t\in[a,b]} \|z_n^{(k)}(t) - z^{(k)}(t)\| \Big) (v_P(t_2) - v_P(t_1))
\end{aligned}$$

Now P being continuous implies that v_P is continuous, and therefore g is also continuous by this last inequality. Since $\|z_n^{(k)}(t) - z^{(k)}(t)\|$ is bounded independently of n, k, and t by Lemma 4.1, the sequence is equicontinuous on $[a,b]$. Then by the Ascoli-Arzela Theorem, the sequence has a uniformly convergent subsequence. Since the sequence converges pointwise and has a uniformly convergent subsequence, the sequence itself converges uniformly.

We rewrite the fourth term on the right-hand side of (4.3) using the the function $I_A(t) = \int_a^t A(s)ds$. Then exactly as done in (4.4), integrate by parts and take the norm to get

$$\begin{aligned}
&\Big\| \int_a^t [A(s) - A_n(s)]\, y^{(k)}(s)\, ds \Big\| \\
&= \Big\| \int_a^t [dI_A(s) - dI_{A_n}(s)]\, y^{(k)}(s)\, ds \Big\| \\
&\le \Big(\sup_{t\in[a,b]} \|I_A(s) - I_{A_n}(s)\| \Big) \bigvee_a^b y^{(k)} + \|I_A(t) - I_{A_n}(t)\|\, \|y^{(k)}(t)\|
\end{aligned} \tag{4.6}$$

which converges uniformly to zero since $\bigvee_a^b y^{(k)}$ and $\|y^{(k)}(t)\|$ are bounded, and since $A_n \to A$ weakly in $L_1([a,b])$ implies that $I_{A_n} \to I_A$ uniformly [10, p.430].

For the last term in (4.3), we have

$$\Big\| \int_a^t A_n(s)\,[y_n^{(k)}(s) - y^{(k)}(s)]\,ds \Big\| \le \Big(\sup_{t\in[a,b]} \|y_n^{(k)}(t) - y^{(k)}(t)\| \Big) \int_a^t \|A_n(s)\|\,ds,$$

Now $\int_a^t \|A_n(s)\|\,ds$ is uniformly bounded since $A_n \to A$ weakly in $L_1([a,b])$ [10, p.430], and $y_n^{(k)}(t) \to y^{(k)}(t)$ uniformly by induction hypothesis. Therefore the term on the right-hand side of (4.6) converges uniformly to zero.

We have shown that for $y^{(k+1)}(t) - y_n^{(k+1)}(t)$ written as the sum of five terms, each term converges to zero uniformly on $[a,b]$, completing the inductive step.

(2) Similarly to part 1, we write

$$\begin{aligned} z^{(k+1)}(t) - z_n^{(k+1)}(t) &= z(a) + \int_a^t [dM(s) y^{(k)}(s) + D(s)\,z^{(k)}(s)\,ds] \\ &\quad - z_n(a) - \int_a^t [dM_n(s)\,y_n^{(k)}(s) + D_n(s)\,z_n^{(k)}(s)\,ds]. \end{aligned}$$

Then

$$\begin{aligned} &z^{(k+1)}(t) - z_n^{(k+1)}(t) \\ &= [z(a) - z_n(a)] + \int_a^t [dM(s) - dM_n(s)]\,y^{(k)}(s) \\ &\quad - \int_a^t dM_n(s)\,[y_n^{(k)}(s) - y^{(k)}(s)] + \int_a^t [D(s) - D_n(s)]\,z^{(k)}(s)\,ds \\ &\quad - \int_a^t D_n(s)\,[z_n^{(k)}(s) - z^{(k)}(s)]\,ds. \end{aligned} \tag{4.7}$$

We will show that each of these terms converges to zero pointwise in $[a,b]$. First, $\|z(a) - z_n(a)\| \to 0$ by assumption.

The pointwise convergence to zero of the second term on the right-hand side of (4.7), $\int_a^t [dM(s) - dM_n(s)]\,y^{(k)}(s)$, follows from Theorem 2.6.

For the third term on the right-hand side of (4.7), we have

$$\Big\| \int_a^t dM_n(s)\,[y_n^{(k)}(s) - y^{(k)}(s)] \Big\| \le \Big(\sup_{t\in[a,b]} \|y_n^{(k)}(t) - y^{(k)}(t)\| \Big) \Big(\sup_n \bigvee_a^b M_n \Big),$$

which converges uniformly to zero since $\bigvee_a^b M_n$ is bounded independently of n and since $y_n^{(k)}(t) \to y^{(k)}(t)$ uniformly by inductive hypothesis.

We rewrite the fourth term on the right-hand side of (4.7) using the function $I_D(t) = \int_a^t D(s)ds$ and the same method of integrating by parts, getting

$$\begin{aligned} &\Big\| \int_a^t [D(s) - D_n(s)]\,z^{(k)}(s)\,ds \Big\| \\ &= \Big\| \int_a^t [dI_D(s) - dI_{D_n}(s)]\,z^{(k)}(s) \Big\| \\ &\le \Big(\sup_{t\in[a,b]} \|I_D(s) - I_{D_n}(s)\| \Big) \bigvee_a^b z^{(k)} + \|I_D(t) - I_{D_n}(t)\|\,\|z^{(k)}(t)\|. \end{aligned}$$

The right-hand side converges uniformly to zero since $\bigvee_a^b z^{(k)}$ and $\|z^{(k)}(t)\|$ are bounded and since $D_n \to D$ weakly in $L_1([a,b])$ implies that $I_{D_n} \to I_D$ uniformly.

We write the last term in (4.7) as

$$\begin{aligned}
&\int_a^t D_n(s)\,[z_n^{(k)}(s) - z^{(k)}(s)]\,ds \\
&= \int_a^t [dI_{D_n}(s) - dI_D(s)]\,[z_n^{(k)}(s) - z^{(k)}(s)] + \int_a^t dI_D(s)\,[z_n^{(k)}(s) - z^{(k)}(s)].
\end{aligned}$$

The first of these two integrals can be shown to converge uniformly to zero using integration by parts as seen before. Theorem 2.7 can be used to show pointwise convergence of the second integral. This theorem applies because $D \in L_1([a,b])$ implies that I_D is of bounded variation, and because $z_n^{(k)}(t) \to z^{(k)}(t)$ pointwise by induction hypothesis. The convergence can be shown to be uniform using the Ascoli-Arzela Theorem as in part 1, i.e., as in the last term of (4.5).

Having shown that $z^{(k+1)}(t) - z_n^{(k+1)}(t)$ can be written as the sum of five terms, one converging pointwise and four converging uniformly in $[a,b]$, the inductive step is complete. □

Corollary 4.3. *The convergence $\|z^{(k)}(t) - z_n^{(k)}(t)\| \to 0$ as $n \to \infty$ is uniform on $[a,b]$ if $M_n \to M$ uniformly as $n \to \infty$ on $[a,b]$, for $k = 1,2,\dots$.*

Proof. In the proof of Lemma 4.2 (2), $z^{(k+1)}(t) - z_n^{(k+1)}(t)$ was written as the sum of five terms, all of which converge uniformly to zero except for the term $\int_a^t [dM(s) - dM_n(s)] y^{(k)}(s)$, which converges pointwise. Under the assumption that $M_n \to M$ uniformly, we now show that this term converges uniformly. Using integration by parts in the same manner as used in the proof of the previous Lemma,

$$\begin{aligned}
&\Big\| \int_a^t [dM(s) - dM_n(s)]\, y^{(k)}(s) \Big\| \\
&\le \Big(\sup_{t\in[a,b]} \|M(t) - M_n(t)\| \Big) \bigvee_a^b y^{(k)} + \|M(t) - M_n(t)\|\, \|y^{(k)(t)}\|,
\end{aligned}$$

which converges uniformly to zero since $\bigvee_a^b y^{(k)}$ and $\|y^{(k)}\|$ are bounded and since $M_n \to M$ uniformly. □

Theorem 4.4. *Let $\begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$ be the solution of* (3.1) *and $\begin{bmatrix} y_n(t) \\ z_n(t) \end{bmatrix}$ be the solution of* (4.1)*, and assume* (4.2) *holds. Then as $n \to \infty$,*

1. *$y_n(t) \to y(t)$ uniformly in $[a,b]$, and*
2. *$z_n(t) \to z(t)$ pointwise in $[a,b]$.*

Proof. Assume (4.2) holds.

(1) We write

$$\|y(t) - y_n(t)\| \le \|y(t) - y^{(k)}(t)\| + \|y^{(k)}(t) - y_n^{(k)}(t)\| + \|y_n^{(k)}(t) - y_n(t)\|.$$

As in the proof of Thoerem 3.5, using Lemma 3.2, we have

$$\Big\| \begin{bmatrix} y_n^{(k)}(t) \\ z_n^{(k)}(t) \end{bmatrix} - \begin{bmatrix} y_n(t) \\ z_n(t) \end{bmatrix} \Big\| \le L_n \sum_{i=k+1}^{\infty} \frac{f_n(b)^i}{i!},$$

which tends to zero as $k \to \infty$. Since the L_n and $f_n(b)$ are bounded independent of n, this convergence is independent of n, and it is clearly independent of t and k as well. So given any $\epsilon > 0$, there exists a $k_0 > 0$ such that if $k \geq k_0$, then

$$\|y_n^{(k)}(t) - y_n(t)\| < \frac{\epsilon}{3}$$

for all $n = 1, 2, 3, \dots.$ and $t \in [a, b]$.

Also, by the convergence of successive approximations to the solution as shown in the proof of Theorem 3.5, we can assume for the same k_0 that $k \geq k_0$ implies

$$\|y(t) - y^{(k)}(t)\| < \frac{\epsilon}{3}$$

for all $n = 1, 2, 3, \dots$ and $t \in [a, b]$. By Lemma 4.2, we can assume that for the same k_0 that $k \geq k_0$ implies

$$\|y_n^{(k)}(t) - y^{(k)}(t)\| < \frac{\epsilon}{3}.$$

Therefore, for $k \geq k_0$, $\|y(t) - y_n(t)\| < \epsilon$ for $n = 1, 2, 3, \dots$ and $t \in [a, b]$.

(2) Write

$$\|z(t) - z_n(t)\| \leq \|z(t) - z^{(k)}(t)\| + \|z^{(k)}(t) - z_n^{(k)}(t)\| + \|z_n^{(k)}(t) - z_n(t)\|,$$

and in a similar manner obtain $\|z(t) - z_n(t)\| < \epsilon$, but this time the convergence of the middle term is pointwise rather than uniform (see Lemma 4.2). □

The next corollary follows directly from Corollary 4.3 and Theorem 4.2 (2).

Corollary 4.5. *The convergence $z_n(t) \to z(t)$ is uniform if $M_n \to M$ is uniform.*

Theorem 4.4 gives convergence of solutions to initial value problems under weak conditions of convergence of the coefficients. In a second work, we will show how this gives the continuous and differentiable dependence of eigenvalues and eigenfunctions on the data–coefficients, boundary conditions, and endpoints.

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