Electronic Journal of Differential Equations, Vol. 2004(2004), No. 89, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

MULTIPLE SIGN-CHANGING SOLUTIONS FOR SOME M-POINT BOUNDARY-VALUE PROBLEMS

XIAN XU

Abstract. In this paper, we show existence results for multiple sign-changing solutions for m-point boundary-value problems. We use fixed point index and Leray-Schauder degree methods.

1. Introduction

In this paper, we consider the second-order multi-point boundary-value problem

$$\begin{gathered} y''(t) + f(y) = 0, \quad 0 \le t \le 1, \\ y(0) = 0, \quad y(1) = \sum_{i=1}^{m-2} \alpha_i y(\eta_i), \end{gathered} \tag{1.1}$$

where $0 < \alpha_i$, $i = 1, 2, \dots, m-2$, $0 < \eta_1 < \eta_2 < \dots < \eta_{m-2} < 1$, $f \in C(\mathbb{R}, \mathbb{R})$.

The multi-point boundary-value problems for ordinary differential equations arise in different areas of applied mathematics and physics. For examples, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities can be set up as a multi-point boundary-value problem (see [11]), many problems in the theory of elastic stability can be handled as multi-point problems (see [13]). Recently, there is much attention focused on the existence of nontrivial or positive solutions of the nonlinear multi-point boundary-value problems(see [3, 4, 5, 7, 9, 10, 12, 14, 15, 16, 17] and the references therein). For example, Ruyun Ma [9] considered the m-point boundary-value problem

$$\begin{gathered} u''(t) + a(t)f(u) = 0, \quad t \in (0,1), \\ u'(0) = \sum_{i=1}^{m-2} b_i u'(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \end{gathered} \tag{1.2}$$

where $f \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $a_i, \ b_i \in \mathbb{R}^+$ with $0 < \sum_{i=1}^{m-2} a_i < 1$, and $0 < \sum_{i=1}^{m-2} b_i < 1$. Set

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \to +\infty} \frac{f(u)}{u}.$$

2000 *Mathematics Subject Classification.* 34B15, 34B25.

Key words and phrases. m-point boundary-value problem; sign-changing solution; fixed point index.

Submitted April 26, 2004. Published July 5, 2004.

Then $f_0 = 0$ and $f_\infty = \infty$ correspond to the super-linear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sub-linear case. By applying the fixed point theorem in cones, Ruyun Ma [9] showed that the m-point boundary value problem (1.2) has at least one positive solution if f is either super-linear or sub-linear.

In this paper, we shall study the cases $f_0,\ f_\infty \notin \{0, +\infty\}$. In these cases, the m-point boundary-value problem (1.1) may have sign-changing solutions. Quite recently, the existence and qualitative properties of sign-changing solutions for elliptic boundary-value problems have been extensively studied. To the author's knowledge, however, there were fewer papers considered the sign-changing solutions for multi-point boundary value problems. The purpose of this paper is to give some existence results for multiple sign-changing solution for m-point boundary value problem (1.1). We shall follow the idea employed in [8] by Liu. To show the main result in this paper we need to study the the spectrum properties of the linear operator related the m-point boundary-value problem (1.1). Gupta and Sergej Trofimchuk [4] studied the problem of existence of solutions for the three-point boundary-value problem

$$\begin{gathered} x''(t) = f(t, x(t), x'(t)), \quad t \in (0,1), \\ x(0) = 0, \quad x(1) = \alpha x(\eta), \end{gathered} \tag{1.3}$$

where $\alpha \in \mathbb{R}$, $\alpha \le 1$ and $\eta \in (0,1)$ are given. Using the spectrum radius of some related linear operators, the authors proved some existence results for nontrivial solutions of the three-point boundary-value problem (1.3).

We shall organize this paper as follows. In §2 some preliminary results are given including the study of the eigenvalues of the linear operator $A'(\theta)$ and $A'(\infty)$. In §3 by using the fixed point index and Leray-Shauder degree method, we will prove the main result.

2. Preliminary Lemmas

From [1, Theorem 2.3.1], we have the following definition. Let X be a retract of real Banach space E, U be a relatively bounded open subset of X, $A : D \mapsto X$ be completely continuous operator. The integer $i(A, U, X)$ be defined by

$$i(A, U, X) = \deg(I - A \cdot r, B(\theta, R) \cap r^{-1}(U), \theta),$$

where $r : E \mapsto X$ is an arbitrary retraction and $R > 0$ such that $B(\theta, R) \supset U$. Then the integer $i(A, U, X)$ be called the fixed point index of A on U with respect to X.

Set

$$\beta_0 = \lim_{x \to 0} \frac{f(x)}{x}, \quad \beta_1 = \lim_{|x| \to \infty} \frac{f(x)}{x}.$$

Let us list some conditions to be used in this paper.

(H0) Assume that the sequence of positive solutions of the equation

$$\sin \sqrt{x} = \sum_{i=1}^{m-2} \alpha_i \sin \eta_i \sqrt{x}$$

is $\lambda_1 < \lambda_2 < \dots < \lambda_n < \lambda_{n+1} < \dots$.

(H1) $0 < \sum_{i=1}^{m-2} \alpha_i < 1$, $f \in C(\mathbb{R}, \mathbb{R})$, $f(0) = 0$, $xf(x) > 0$ for all $x \in \mathbb{R} \backslash \{0\}$.

(H2) There exist positive integers n_0 and n_1 such that

$$\lambda_{2n_0} < \beta_0 < \lambda_{2n_0+1}, \quad \lambda_{2n_1} < \beta_1 < \lambda_{2n_1+1}.$$

(H3) There exists $C_0 > 0$ such that
$$|f(x)| < \frac{2(1-\sum_{i=1}^{m-2}\alpha_i\eta_i)}{5-\sum_{i=1}^{m-2}\alpha_i\eta_i}C_0,$$
for all x with $|x| \le C_0$.

The main result of this paper is the following.

Theorem 2.1. *Suppose that (H0)–(H3) hold. Then the m-point boundary-value problem* (1.1) *has at least two sign-changing solutions. Moreover, the m-point boundary-value problem* (1.1) *also has at least two positive solutions and two negative solutions.*

Before giving the proof of Theorem 2.1, we list some preliminary lemmas. Let
$$E = \{x \in C^1[0,1] : x(0) = 0,\ x(1) = \sum_{i=1}^{m-2}\alpha_i x(\eta_i)\}$$
$$P = \{x \in E : x(t) \ge 0 \text{ for } t \in [0,1]\}.$$
For $x \in E$, let $\|x\| = \|x\|_0 + \|x'\|_0$, where $\|x\|_0 = \max_{t\in[0,1]}|x(t)|$ and $\|x'\|_0 = \max_{t\in[0,1]}|x'(t)|$. It is easy to show that E is a Banach space with the norm $\|\cdot\|$ and P is a cone of E. Let the operators K, F and A be defined by
$$\begin{aligned}(Kx)(t) = \frac{t}{1-\sum_{i=1}^{m-2}\alpha_i\eta_i}\int_0^1(1-s)x(s)ds - \int_0^t(t-s)x(s)ds \\ - \frac{t}{1-\sum_{i=1}^{m-2}\alpha_i\eta_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{\eta_i}(\eta_i-s)x(s)ds, \quad t\in[0,1],\ x\in E,\end{aligned} \tag{2.1}$$
$(Fx)(t) = f(x(t))$ for $t \in [0,1]$, $x \in E$ and $A = KF$.

From [1, Lemma 2.3.1], we get the following Lemma.

Lemma 2.2. *Let $\theta \in \Omega$ and $A : P\cap\bar{\Omega} \mapsto P$ be condensing. Suppose that*
$$Ax \ne \mu x, \quad \forall x \in P\cap\partial\Omega,\ \mu \ge 1.$$
Then $i(A, P\cap\Omega, P) = 1$.

From [2, Corollary 2, p.p.146], we have the following Lemma.

Lemma 2.3. *Let Ω be a open set in E and $\theta \in \Omega$, $A : \bar{\Omega} \mapsto E$ be completely continuous. Suppose that*
$$\|Ax\| \le \|x\|, \quad Ax \ne x,\ \forall x \in \partial\Omega.$$
Then $\deg(I - A, \Omega, \theta) = 1$.

Remark Obviously, Lemma 2.3 can also be directly obtained by the normality and homotopic invariance property of Leray-Schauder degree.

The following Lemma can be easily obtained.

Lemma 2.4. *Suppose that $\sum_{i=1}^{m-2}\alpha_i\eta_i < 1$. If $u \in C[0,1]$, then $y \in C^2[0,1]$ is a solution the m-point boundary-value problem*
$$y''(t) + u(t) = 0, \quad 0 \le t \le 1,$$
$$y(0) = 0, \quad y(1) = \sum_{i=1}^{m-2}\alpha_i y(\eta_i)$$

if and only if $y \in C[0,1]$ *is a solution of the integral equation* $y(t) = (Ku)(t), t \in [0,1]$.

Remark By Lemma 2.4 we can easily show that $A : E \mapsto E$ is a completely continuous operator.

Lemma 2.5. *Suppose that (H1) and (H2) hold. Then the operator* A *is Fréchet differentiable at* θ *and* ∞. *Moreover,* $A'(\theta) = \beta_0 K$, *and* $A'(\infty) = \beta_1 K$.

Proof. For any $\varepsilon > 0$, by (H2) there exists $\delta > 0$ such that for any $0 < |x| < \delta$,

$$\Big|\frac{f(x)}{x} - \beta_0\Big| < \varepsilon,$$

that is $|f(x) - \beta_0 x| < \varepsilon|x|$, for all $0 \le |x| < \delta$. Then, for any $x \in E$ with $\|x\| < \delta$, we have

$$\begin{aligned}
&|(Ax - A\theta - \beta_0 Kx)(t)| \\
&= |(K(Fx - \beta_0 x))(t)| \\
&\le \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \int_0^1 (1-s) \max_{s\in[0,1]} |f(x(s)) - \beta_0 x(s)| ds \\
&\quad + \int_0^1 (1-s) \max_{s\in[0,1]} |f(x(s)) - \beta_0 x(s)| ds \\
&\quad + \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} (\eta_i - s) \max_{s\in[0,1]} |f(x(s)) - \beta_0 x(s)| ds \\
&\le \Big[\frac{1}{2(1 - \sum_{i=1}^{m-2} \alpha_i \eta_i)} + \frac{1}{2} + \frac{\sum_{i=1}^{m-2} \alpha_i \eta_i^2}{2(1 - \sum_{i=1}^{m-2} \alpha_i \eta_i)}\Big] \|x\|_0 \varepsilon \\
&\le \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \|x\| \varepsilon, t \in [0,1].
\end{aligned}$$

This implies

$$\|Ax - A\theta - \beta_0 Kx\|_0 \le \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \|x\| \varepsilon, \quad x \in E,\ \|x\| < \delta. \tag{2.2}$$

Similarly, we can show that for any $x \in E$, $\|x\| < \delta$,

$$|(Ax - A\theta - \beta_0 Kx)'(t)| \le \frac{3 - \sum_{i=1}^{m-2} \alpha_i \eta_i}{2(1 - \sum_{i=1}^{m-2} \alpha_i \eta_i)} \|x\| \varepsilon, \quad t \in [0,1]$$

and so

$$\|(Ax - A\theta - \beta_0 Kx)'\|_0 \le \frac{3 - \sum_{i=1}^{m-2} \alpha_i \eta_i}{2(1 - \sum_{i=1}^{m-2} \alpha_i \eta_i)} \|x\| \varepsilon,\ x \in E,\ \|x\| < \delta. \tag{2.3}$$

By (2.2) and (2.3), we have

$$\begin{aligned}
\|Ax - A\theta - \beta_0 Kx\| &= \|Ax - A\theta - \beta_0 Kx\|_0 + \|(Ax - A\theta - \beta_0 Kx)'\|_0 \\
&\le \frac{5 - \sum_{i=1}^{m-2} \alpha_i \eta_i}{2(1 - \sum_{i=1}^{m-2} \alpha_i \eta_i)} \|x\| \varepsilon
\end{aligned}$$

Consequently,

$$\lim_{\|x\|\to 0} \frac{\|Ax - A\theta - \beta_0 Kx\|}{\|x\|} = 0.$$

This means that A is Fréchet differentiable at θ, and $A'(\theta) = \beta_0 K$.

For each $\varepsilon > 0$, by (H2), there exists $R > 0$ such that

$$|f(x) - \beta_1 x| < \varepsilon |x|$$

for $|x| > R$. Let $b = \max_{|x| \le R} |f(x) - \beta_1 x|$. Then we have for any $x \in \mathbb{R}$,

$$|f(x) - \beta_1 x| \le \varepsilon |x| + b.$$

Consequently,

$$\begin{aligned}
&|(Ax - \beta_1 Kx)(t)| \\
&= |(K(Fx - \beta_1 x))(t)| \\
&\le \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \int_0^1 (1-s) \max_{s \in [0,1]} |f(x(s)) - \beta_1 x(s)| ds \\
&\quad + \int_0^1 (1-s) \max_{s \in [0,1]} |f(x(s)) - \beta_1 x(s)| ds \\
&\quad + \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} (\eta_i - s) \max_{s \in [0,1]} |f(x(s)) - \beta_1 x(s)| ds \\
&\le \Big[\frac{1}{2(1 - \sum_{i=1}^{m-2} \alpha_i \eta_i)} + \frac{1}{2} + \frac{\sum_{i=1}^{m-2} \alpha_i \eta_i^2}{2(1 - \sum_{i=1}^{m-2} \alpha_i \eta_i)} \Big] (\varepsilon \|x\|_0 + b) \\
&\le \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} (\varepsilon \|x\| + b), \quad t \in [0,1].
\end{aligned}$$

This implies

$$\|Ax - \beta_1 Kx\|_0 \le \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} (\varepsilon \|x\| + b), \quad x \in E. \tag{2.4}$$

Similarly, we can show that

$$\|(Ax - \beta_1 Kx)'\|_0 \le \frac{3 - \sum_{i=1}^{m-2} \alpha_i \eta_i}{2(1 - \sum_{i=1}^{m-2} \alpha_i \eta_i)} (\varepsilon \|x\| + b), \quad x \in E. \tag{2.5}$$

By (2.4) and (2.5), we have

$$\begin{aligned}
\|Ax - \beta_1 Kx\| &= \|Ax - \beta_1 Kx\|_0 + \|(Ax - \beta_1 Kx)'\|_0 \\
&\le \frac{5 - \sum_{i=1}^{m-2} \alpha_i \eta_i}{2(1 - \sum_{i=1}^{m-2} \alpha_i \eta_i)} (\varepsilon \|x\| + b).
\end{aligned}$$

Consequently,

$$\lim_{\|x\| \to \infty} \frac{\|Ax - \beta_1 Kx\|}{\|x\|} = 0\,.$$

This means that A is Fréchet differentiable at ∞, and $A'(\infty) = \beta_1 K$. The proof is complete. □

Lemma 2.6. *Suppose that (H0) and (H1) hold. Let β be a positive number. Then the sequence of positive eigenvalues of the operator βK is*

$$\frac{\beta}{\lambda_1} > \frac{\beta}{\lambda_2} > \dots > \frac{\beta}{\lambda_n} \dots .$$

Moreover, the positive eigenvalues $\frac{\beta}{\lambda_n}$ $(n = 1, 2, \dots)$ have algebraic multiplicity one.

Proof. Let $\bar{\lambda}$ be a positive eigenvalue of the linear operator βK, and $y \in E\backslash\{\theta\}$ be an eigenfunction corresponding to the eigenvalue $\bar{\lambda}$. By Lemma 2.4, we have

$$\begin{gathered} y''(t) + \frac{\beta}{\bar{\lambda}} y(t) = 0, 0 \le t \le 1, \\ y(0) = 0, \quad y(1) = \sum_{i=1}^{m-2} \alpha_i y(\eta_i). \end{gathered} \tag{2.6}$$

The auxiliary equation of the differential equation (2.6) has roots $\pm\sqrt{\frac{\beta}{\bar{\lambda}}}i$. Thus the general solution of (2.6) is of the form

$$y(t) = C_1 \cos t\sqrt{\frac{\beta}{\bar{\lambda}}} + C_2 \sin t\sqrt{\frac{\beta}{\bar{\lambda}}}, \quad t \in [0,1].$$

Applying the condition $y(0) = 0$, we obtain that $C_1 = 0$, and so the general solution can be reduce to

$$y(t) = C_2 \sin t\sqrt{\frac{\beta}{\bar{\lambda}}}, \quad t \in [0,1].$$

Applying the second condition $y(1) = \sum_{i=1}^{m-2} \alpha_i y(\eta_i)$, we obtain that

$$\sin\sqrt{\frac{\beta}{\bar{\lambda}}} = \sum_{i=1}^{m-2} \alpha_i \sin \eta_i \sqrt{\frac{\beta}{\bar{\lambda}}}.$$

Since the positive solutions of the equation $\sin\sqrt{x} = \sum_{i=1}^{m-2} \alpha_i \sin \eta_i \sqrt{x}$ are $0 < \lambda_1 < \lambda_2 < \dots$, then $\bar{\lambda}$ is one of the values

$$\frac{\beta}{\lambda_1} > \frac{\beta}{\lambda_2} > \dots > \frac{\beta}{\lambda_n} \dots$$

and the eigenfunction corresponding to the eigenvalue $\frac{\beta}{\lambda_n}$ is

$$y_n(t) = C \sin t\sqrt{\lambda_n}, \quad t \in [0,1],$$

where C is a nonzero constant. By ordinary method, we can show that any two eigenfunctions corresponding to the same eigenvalue $\frac{\beta}{\lambda_n}$ are merely nonzero constant multiples of each other. Consequently,

$$\dim\ker(\frac{\beta}{\lambda_n} I - \beta K) = \dim\ker(I - \lambda_n K) = 1. \tag{2.7}$$

Now we show that

$$\ker(I - \lambda_n K) = \ker(I - \lambda_n K)^2. \tag{2.8}$$

Obviously, we need to show only that

$$\ker(I - \lambda_n K)^2 \subset \ker(I - \lambda_n K).$$

For any $y \in \ker(I - \lambda_n K)^2$, $(I - \lambda_n K)y$ is an eigenfunction of linear operator βK corresponding to the eigenvalue $\frac{\beta}{\lambda_n}$ if $(I - \lambda_n K)y \neq \theta$. Then there exists nonzero constant γ such that

$$(I - \lambda_n K)y = \gamma \sin t\sqrt{\lambda_n}, \quad t \in [0,1].$$

By direct computation, we have

$$\begin{gathered}
y''(t)+\lambda_n y=-\lambda_n\gamma\sin t\sqrt{\lambda_n},\quad t\in[0,1],\\
y(0)=0,\quad y(1)=\sum_{i=1}^{m-2}\alpha_i y(\eta_i).
\end{gathered}\tag{2.9}$$

It is easy to see that the general solutions of (2.9) is of the form

$$\begin{aligned}
y(t)&=C_1\cos t\sqrt{\lambda_n}+C_2\sin t\sqrt{\lambda_n}+(\frac{\gamma t\sqrt{\lambda_n}}{2}-\frac{\gamma}{4}\sin 2t\sqrt{\lambda_n})\cos t\sqrt{\lambda_n}\\
&\quad+\frac{\gamma}{4}\cos 2t\sqrt{\lambda_n}\cdot\sin t\sqrt{\lambda_n},\quad t\in[0,1],
\end{aligned}$$

where C_1, C_2 are two nonzero constants. Applying the condition $y(0)=0$, we obtain that $C_1=0$. Since $\sin\sqrt{\lambda_n}=\sum_{i=1}^{m-2}\alpha_i\sin\eta_i\sqrt{\lambda_n}$, then we have

$$\begin{aligned}
y(1)&=C_2\sin\sqrt{\lambda_n}+(\frac{\gamma\sqrt{\lambda_n}}{2}-\frac{\gamma}{4}\sin 2\sqrt{\lambda_n})\cos\sqrt{\lambda_n}+\frac{\gamma}{4}\cos 2\sqrt{\lambda_n}\cdot\sin\sqrt{\lambda_n}\\
&=\sum_{i=1}^{m-2}\alpha_i C_2\sin\eta_i\sqrt{\lambda_n}+\frac{\gamma\sqrt{\lambda_n}}{2}\cos\sqrt{\lambda_n}-\frac{\gamma}{2}\sum_{i=1}^{m-2}\alpha_i\sin\eta_i\sqrt{\lambda_n}\cos^2\sqrt{\lambda_n}\\
&\quad+\frac{\gamma}{4}\sum_{i=1}^{m-2}\alpha_i\cos 2\sqrt{\lambda_n}\sin\eta_i\sqrt{\lambda_n},
\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}
\sum_{i=1}^{m-2}\alpha_i y(\eta_i)&=\sum_{i=1}^{m-2}\alpha_i C_2\sin\eta_i\sqrt{\lambda_n}+\sum_{i=1}^{m-2}(\frac{\gamma\alpha_i\eta_i\sqrt{\lambda_n}}{2}-\frac{\gamma\alpha_i}{4}\sin 2\eta_i\sqrt{\lambda_n})\cos\eta_i\sqrt{\lambda_n}\\
&\quad+\sum_{i=1}^{m-2}\frac{\gamma\alpha_i}{4}\cos 2\eta_i\sqrt{\lambda_n}\cdot\sin\eta_i\sqrt{\lambda_n}.
\end{aligned}\tag{2.11}$$

Since $y(1)=\sum_{i=1}^{m-2}\alpha_i y(\eta_i)$, by (2.10) and (2.11), we have

$$\cos\sqrt{\lambda_n}=\sum_{i=1}^{m-2}\alpha_i\eta_i\cos\eta_i\sqrt{\lambda_n}.$$

By the Schwarz inequality, we obtain

$$\begin{aligned}
1-\sin^2\sqrt{\lambda_n}&=(\sum_{i=1}^{m-2}\alpha_i\eta_i\cos\eta_i\sqrt{\lambda_n})^2\\
&\le(\sum_{i=1}^{m-2}\eta_i^2)(\sum_{i=1}^{m-2}\alpha_i^2\cos^2\eta_i\sqrt{\lambda_n})\\
&=(\sum_{i=1}^{m-2}\eta_i^2)(\sum_{i=1}^{m-2}\alpha_i^2)-(\sum_{i=1}^{m-2}\eta_i^2)(\sum_{i=1}^{m-2}\alpha_i^2\sin^2\eta_i\sqrt{\lambda_n}).
\end{aligned}$$

Applying the condition $\sin\sqrt{\lambda_n} = \sum_{i=1}^{m-2} \alpha_i \sin\eta_i\sqrt{\lambda_n}$, we obtain

$$
\begin{aligned}
1 &\le (\sum_{i=1}^{m-2} \eta_i^2)(\sum_{i=1}^{m-2} \alpha_i^2) + (\sum_{i=1}^{m-2} \alpha_i \sin\eta_i\sqrt{\lambda_n})^2 - (\sum_{i=1}^{m-2} \eta_i^2)(\sum_{i=1}^{m-2} \alpha_i^2 \sin^2\eta_i\sqrt{\lambda_n}) \\
&= (\sum_{i=1}^{m-2} \eta_i^2)(\sum_{i=1}^{m-2} \alpha_i^2) + (1 - (\sum_{i=1}^{m-2} \eta_i^2))(\sum_{i=1}^{m-2} \alpha_i^2 \sin^2\eta_i\sqrt{\lambda_n}) \\
&+ \sum_{i\neq j} \alpha_i\alpha_j \sin\eta_i\sqrt{\lambda_n}\sin\eta_j\sqrt{\lambda_n} \\
&\le (\sum_{i=1}^{m-2} \eta_i^2)(\sum_{i=1}^{m-2} \alpha_i^2) + (1 - (\sum_{i=1}^{m-2} \eta_i^2))(\sum_{i=1}^{m-2} \alpha_i^2) + \sum_{i\neq j} \alpha_i\alpha_j \\
&= (\sum_{i=1}^{m-2} \alpha_i)^2,
\end{aligned}
$$

which is a contradiction of $\sum_{i=1}^{m-2} \alpha_i < 1$. Thus, (2.8) holds. It follows from (2.7) and (2.8) that the algebraic multiplicity of the eigenvalue $\frac{\beta}{\lambda_n}$ is 1. The proof is complete. ☐

Lemma 2.7. *Suppose that (H0) and (H1) hold and $y \in P\backslash\{\theta\}$ is a solution of the boundary-value problem* (1.1). *Then $y \in \overset{\circ}{P}$.*

Proof. Since $y''(t) = -f(y(t)) \le 0$ for $t \in [0,1]$, then y is a concave function on $[0,1]$. For all $i \in \{1,2,\dots,m-2\}$, we have from the concavity of y that

$$
y(t) \le \frac{y(1) - y(\eta_i)}{1-\eta_i}(t-1) + y(1), \quad t \in [0,\eta_1]
$$

that is $y(t)(1-\eta_i) \le (y(1) - y(\eta_i))(t-1) + y(1)(1-\eta_i)$, $t \in [0,\eta_1]$. This together with the boundary condition $y(1) = \sum_{i=1}^{m-2} \alpha_i y(\eta_i)$ implies

$$
\begin{aligned}
y(t) &\le y(1)\frac{\sum_{i=1}^{m-2} \alpha_i(1-\eta_i) + (1 - \sum_{i=1}^{m-2} \alpha_i)(1-t)}{\sum_{i=1}^{m-2} \alpha_i(1-\eta_i)} \\
&\le y(1)\frac{\sum_{i=1}^{m-2} \alpha_i(1-\eta_i) + (1 - \sum_{i=1}^{m-2} \alpha_i)}{\sum_{i=1}^{m-2} \alpha_i(1-\eta_i)} \qquad (2.12)\\
&= y(1)\frac{1 - \sum_{i=1}^{m-2} \alpha_i\eta_i}{\sum_{i=1}^{m-2} \alpha_i(1-\eta_i)}, \quad t \in [0,\eta_1].
\end{aligned}
$$

From the concavity of y and this inequality, we have

$$
y(t) \le \frac{y(\eta_1)}{\eta_1}t \le \frac{y(\eta_1)}{\eta_1} \le y(1)\frac{1 - \sum_{i=1}^{m-2} \alpha_i\eta_i}{\sum_{i=1}^{m-2} \alpha_i(1-\eta_i)\eta_1}, t \in [\eta_1, 1]. \qquad (2.13)
$$

From this inequality and (2.12) it follows that

$$
y(1) \ge \frac{\sum_{i=1}^{m-2} \alpha_i(1-\eta_i)\eta_1}{1 - \sum_{i=1}^{m-2} \alpha_i\eta_i}\|y\|_0.
$$

Since y is a concave function on [0,1], we have

$$
y(t) \ge (y(1) - y(0))t = y(1)t \ge \frac{\sum_{i=1}^{m-2} \alpha_i(1-\eta_i)\eta_1}{1 - \sum_{i=1}^{m-2} \alpha_i\eta_i}\|y\|_0 t, \quad t \in [0,1]. \qquad (2.14)
$$

Consequently,

$$y'(0) = \lim_{t\to 0} \frac{y(t)}{t} \geq \frac{\sum_{i=1}^{m-2} \alpha_i (1-\eta_i)\eta_1}{1-\sum_{i=1}^{m-2} \alpha_i \eta_i} \|y\|_0 > 0.$$

Then there exist $\varepsilon > 0$ and $\tau_1 > 0$ such that

$$y'(t) > \tau_1, \forall t \in [0, \varepsilon]. \tag{2.15}$$

By (2.14), there exists $\tau_2 > 0$ such that

$$y(t) > \tau_2, \quad \forall t \in [\varepsilon, 1] \tag{2.16}$$

Let $\tau = \min\{\tau_1, \tau_2\}$. Then by (2.15) and (2.16), we obtain $u(t) \geq 0$, $t \in [0,1]$ for any $u \in E$ with $\|u - y\| < \tau$. Therefore, $B(y,\tau) \subset P$ and $y \in \overset{\circ}{P}$, where $B(y,\tau) = \{x \in E : \|x - y\| < \tau\}$. The proof is complete. □

By [1, Lemmas 2.3.7, 2.3.8], we have the following Lemma.

Lemma 2.8. *Let $A : P \mapsto P$ be completely continuous, Suppose that A is differentiable at θ and ∞ along P and 1 is not an eigenvalue of $A'_+(\theta)$ and $A'_+(\infty)$ corresponding to a positive eigenfunction.*

1. *If $A'_+(\theta)$ has a positive eigenfunction corresponding to an eigenvalue greater than 1, and $A\theta = \theta$. Then there exists $\tau > 0$ such that $i(A, P \cap B(\theta, r), P) = 0$ for any $0 < r < \tau$.*
2. *If $A'_+(\infty)$ has a positive eigenfunction which corresponds to an eigenvalue greater than 1. Then there exists $\varsigma > 0$ such that $i(A, P \cap B(\theta, R), P) = 0$ for any $R > \varsigma$.*

Lemma 2.9. *Suppose that (H0)–(H3) hold. Then*

1. *There exists $C_0 > r_0 > 0$ such that for any $0 < r \leq r_0$,*
$$i(A, P \cap B(\theta, r), P) = 0, \quad i(A, -P \cap B(\theta, r), -P) = 0$$
2. *There exists $R_0 > C_0$ such that for any $R \geq R_0$,*
$$i(A, P \cap B(\theta, R), P) = 0, \quad i(A, -P \cap B(\theta, R), -P) = 0.$$

Proof. We prove only conclusion (1). The same way, conclusion (2) can be proved. First we claim that $K(P) \subset P$ and $K(-P) \subset -P$. Let $x \in P$ be fixed and $y = Kx$. Obviously, $y \in C^1[0,1]$. By direct computation, we have

$$\begin{aligned} y(1) &= \frac{1}{1-\sum_{i=1}^{m-2} \alpha_i \eta_i} \Big(\sum_{i=1}^{m-2} \eta_i \int_0^1 (1-s)x(s)ds - \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} (\eta_i - s)x(s)ds\Big) \\ &\geq \frac{1}{1-\sum_{i=1}^{m-2} \alpha_i \eta_i} \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} (1-\eta_i)sx(s)\,ds \geq 0\,. \end{aligned} \tag{2.17}$$

It follows from Lemma 2.4 that

$$y''(t) = -x(t) \leq 0, \quad \forall t \in [0,1]. \tag{2.18}$$

$$y(0) = 0, \quad y(1) = \sum_{i=1}^{m-2} \alpha_i y(\eta_i) \tag{2.19}$$

By (2.18), we see that y is a concave function on [0,1]. Then the boundary condition (2.17) and (2.19) mean that $y(t) \geq 0$ for $t \in [0,1]$. Therefore, $y \in P$, and so

$K(P) \subset P$, $K(-P) \subset (-P)$. Since $xf(x) > 0$ for $x \in \mathbb{R}\backslash\{0\}$, then we see that $A(P) \subset P$ and $A(-P) \subset (-P)$.

It follows from Lemmas 2.5 and 2.6 that $A'_+(\theta) = \beta_0 K$, β_0/λ_1 (> 1) is an eigenvalue of the linear operator $\beta_0 K$ and the eigenfunction corresponding to $\frac{\beta_0}{\lambda_1}$ is

$$y(t) = C \sin t\sqrt{\lambda_1}, \quad t \in [0, 1],$$

where C is an arbitrary positive constant and λ_1 is the smallest positive solution of the equation $\sin\sqrt{x} = \sum_{i=1}^{m-2} \alpha_i \sin\eta_i\sqrt{x}$. Since

$$\lim_{x\to 0} \frac{\sin\sqrt{x} - \sum_{i=1}^{m-2} \alpha_i \sin\eta_i\sqrt{x}}{\sqrt{x}} = 1 - \sum_{i=1}^{m-2} \alpha_i\eta_i > 0\,,$$

there exists $\delta_0 \in (0, 1)$ small enough such that

$$\frac{\sin\sqrt{\delta_0} - \sum_{i=1}^{m-2} \alpha_i \sin\eta_i\sqrt{\delta_0}}{\sqrt{\delta_0}} \geq \frac{1}{4}(1 - \sum_{i=1}^{m-2} \alpha_i\eta_i) > 0.$$

On the other hand,

$$\sin\sqrt{\pi^2} - \sum_{i=1}^{m-2} \alpha_i \sin\eta_i\sqrt{\pi^2} = -\sum_{i=1}^{m-2} \alpha_i \sin\eta_i\pi < 0.$$

Then, by the intermediate-value principle, $\lambda_1 \in (\delta_0, \pi^2)$. Consequently,

$$y(t) = C \sin t\sqrt{\lambda_1} \geq 0, \quad t \in [0, 1].$$

It follows from Lemma 2.8 that there exists $\tau_0 > 0$ such that $i(A, P \cap B(\theta, r), P) = 0$ for any $0 < r \leq \tau_0$.

Similarly, we can show that there exists $\tau_1 > 0$ such that $i(A, -P \cap B(\theta, r), -P) = 0$ for any $0 < r \leq \tau_1$. Let $r_0 = \min\{\tau_0, \tau_1\}$. Then the conclusion (1) holds and the the proof is complete. □

From [6, Theorems 21.6, 21.2], we have the following two lemmas.

Lemma 2.10. *Let A be a completely continuous operator, let $x_0 \in E$ be a fixed point of A and assume that A is defined in a neighborhood of x_0 and Fréchet differentiable at x_0. If 1 is not an eigenvalue of the linear operator $A'(x_0)$, then x_0 is an isolated singular point of the completely continuous vector field $I - A$ and for small enough $r > 0$*

$$\deg(I - A, B(x_0, r), \theta) = (-1)^k,$$

where k is the sum of the algebraic multiplicities of the real eigenvalues of $A'(x_0)$ in $(1, +\infty)$.

Lemma 2.11. *Let A be a completely continuous operator which is defined on all E. Assume that 1 is not an eigenvalue of the asymptotic derivative. The completely continuous vector field $I - A$ is then nonsingular on spheres $S_\rho = \{x | \|x\| = \rho\}$ of sufficiently large radius ρ and*

$$\deg(I - A, B(\theta, \rho), \theta) = (-1)^k,$$

where k is the sum of the algebraic multiplicities of the real eigenvalues of $A'(\infty)$ in $(1, +\infty)$.

3. Proof of main Theorem

Proof of Theorem 2.1. From Lemma 2.4, a function y is a solution of the boundary-value problem (1.1) if and only if y is a fixed point of the operator A. By (H3), we have for any $x \in E$, $\|x\| = C_0$,

$$
\begin{aligned}
&|(Ax)(t)| \\
&\le \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i\eta_i}\int_0^1 (1-s)\max_{s\in[0,1]}|f(x(s))|ds + \int_0^1 (1-s)\max_{s\in[0,1]}|f(x(s))|ds \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i\eta_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{\eta_i}(\eta_i - s)\max_{s\in[0,1]}|f(x(s))|ds \\
&< \frac{2(1-\sum_{i=1}^{m-2}\alpha_i\eta_i)}{5-\sum_{i=1}^{m-2}\alpha_i\eta_i}\Big(\frac{1}{2(1-\sum_{i=1}^{m-2}\alpha_i\eta_i)} + \frac{1}{2} + \frac{\sum_{i=1}^{m-2}\alpha_i\eta_i}{2(1-\sum_{i=1}^{m-2}\alpha_i\eta_i)}\Big)C_0 \\
&\le \frac{2C_0}{5-\sum_{i=1}^{m-2}\alpha_i\eta_i}, \quad t \in [0,1]\,.
\end{aligned}
$$

Therefore,

$$
\|Ax\|_0 < \frac{2C_0}{5-\sum_{i=1}^{m-2}\alpha_i\eta_i}. \tag{3.1}
$$

Similarly, we can show that for any $x \in E$, with $\|x\| = C_0$,

$$
\|(Ax)'\|_0 < \frac{3-\sum_{i=1}^{m-2}\alpha_i\eta_i}{5-\sum_{i=1}^{m-2}\alpha_i\eta_i}C_0. \tag{3.2}
$$

It follows from (3.1) and (3.2) that $\|Ax\| < C_0$, for all $\|x\| = C_0$. Then, by Lemmas 2.2 and 2.3 we have

$$
i(A, P \cap B(\theta, C_0), P) = 1, \tag{3.3}
$$

$$
i(A, -P \cap B(\theta, C_0), -P) = 1, \tag{3.4}
$$

$$
\deg(I - A, B(\theta, C_0), \theta) = 1. \tag{3.5}
$$

By (H2) and Lemma 2.6, the eigenvalues of the operator $A'(\theta) = \beta_0 K$ which are large than 1 are

$$
\frac{\beta_0}{\lambda_1}, \quad \frac{\beta_0}{\lambda_2}, \quad \frac{\beta_0}{\lambda_3}, \dots, \frac{\beta_0}{\lambda_{2n_0}}.
$$

Therefore, by Lemmas 2.6, 2.9, and 2.10, there exists $0 < r_1 < r_0$ such that

$$
\deg(I - A, B(\theta, r_1), \theta) = (-1)^{2n_0} = 1\,. \tag{3.6}
$$

Similarly, by Lemmas 2.6, 2.9 and 2.11, we have for some $R_1 \ge R_0$,

$$
\deg(I - A, B(\theta, R_1), \theta) = 1\,. \tag{3.7}
$$

By Lemma 2.9, we have

$$
i(A, P \cap B(\theta, r_1), P) = 0, \tag{3.8}
$$

$$
i(A, -P \cap B(\theta, r_1), -P) = 0, \tag{3.9}
$$

$$
i(A, P \cap B(\theta, R_1), P) = 0, \tag{3.10}
$$

$$i(A, -P \cap B(\theta, R_1), -P) = 0. \tag{3.11}$$

Then, by (3.3), (3.8) and (3.10), we have

$$i(A, P \cap (B(\theta, R_1)\backslash\overline{B(\theta, C_0)}), P) = 0 - 1 = -1, \tag{3.12}$$

$$i(A, P \cap (B(\theta, C_0)\backslash\overline{B(\theta, r_1)}), P) = 1 - 0 = 1. \tag{3.13}$$

Therefore, the operator A has at least two fixed points $x_1 \in P\cap(B(\theta, R_1)\backslash\overline{B(\theta, C_0)})$ and $x_2 \in P \cap (B(\theta, C_0)\backslash\overline{B(\theta, r_1)})$, respectively. Obviously, x_1 and x_2 are positive solutions of the boundary-value problem (1.1).

Similarly, by (3.4), (3.9) and (3.11), we have

$$i(A, -P \cap (B(\theta, R_1)\backslash\overline{B(\theta, C_0)}), -P) = -1, \tag{3.14}$$

$$i(A, -P \cap (B(\theta, C_0)\backslash\overline{B(\theta, r_1)}), -P) = 1. \tag{3.15}$$

Therefore, the operator A has at least two fixed points $x_3 \in (-P)\cap(B(\theta, C_0)\backslash\overline{B(\theta, r_1)})$ and $x_4 \in (-P) \cap (B(\theta, R_1)\backslash\overline{B(\theta, C_0)})$, respectively. Obviously, x_3 and x_4 are negative solutions of the boundary-value problem (1.1).

Let

$$S = \{x|x = Ax, x \in P \cap (B(\theta, R_1)\backslash\overline{B(\theta, C_0)})\}.$$

It follows from Lemma 2.7 that $S \subset \overset{\circ}{P}$. Therefore, for any $x \in S$, there exists $\delta_x > 0$ such that $B(x, \delta_x) \subset P \cap (B(\theta, R_1)\backslash\overline{B(\theta, C_0)})$. Let $O_1 = \bigcup_{x\in S} B(x, \delta_x)$. Then, we have $O_1 \subset P \cap (B(\theta, R_1)\backslash\overline{B(\theta, C_0)})$. By (3.12) and the excision property of the fixed point index, we have

$$i(A, O_1, P) = -1. \tag{3.16}$$

By the definition of the fixed point index, we have

$$i(A, O_1, P) = \deg(I - A \cdot r, B(\theta, \bar{R}) \cap r^{-1}(O_1), \theta), \tag{3.17}$$

where $r : E \mapsto P$ is an arbitrary retraction and $\bar{R}$ is a large enough positive number such that $O_1 \subset B(\theta, \bar{R})$. Now, we assume that $y^* \in B(\theta, \bar{R}) \cap r^{-1}(O_1)$ such that $y^* = A\cdot r(y^*)$. Since $r : E \mapsto P$ and $A : P \mapsto P$, then $y^* \in P$, and so $y^* = ry^* \in O_1$. Therefore, $y^* \in O_1$ whenever $y^* \in B(\theta, \bar{R})\cap r^{-1}(O_1)$ is a fixed point of the operator $A \cdot r$. Then, by the excision property of the degree we have

$$\deg(I - A \cdot r, B(\theta, \bar{R}) \cap r^{-1}(O_1), \theta) = \deg(I - A, O_1, \theta). \tag{3.18}$$

By (3.16)-(3.18), we have

$$\deg(I - A, O_1, \theta) = -1. \tag{3.19}$$

Similarly, by (3.13)-(3.15), we can show that there exist open sets O_2, O_3 and O_4 such that

$$O_2 \subset P \cap (B(\theta, C_0)\backslash\overline{B(\theta, r_1)}),$$

$$O_3 \subset -P \cap (B(\theta, C_0)\backslash\overline{B(\theta, r_1)}),$$

$$O_4 \subset -P \cap (B(\theta, R_1)\backslash\overline{B(\theta, C_0)}),$$

$$\deg(I - A, O_2, \theta) = 1, \tag{3.20}$$

$$\deg(I - A, O_3, \theta) = 1, \tag{3.21}$$

$$\deg(I - A, O_4, \theta) = -1. \tag{3.22}$$

It follows from (3.5), (3.6), (3.20) and (3.21) that

$$\deg(I-A, B(\theta, C_0)\backslash(\overline{O_2}\cup\overline{O_3}\cup\overline{B(\theta, r_1)}), \theta) = 1-1-1-1 = -2.$$

This implies that A has at least one fixed point $x_5 \in B(\theta, C_0)\backslash(\overline{O_2}\cup\overline{O_3}\cup\overline{B(\theta, r_1)})$. Similarly, by (3.5), (3.7), (3.19) and (3.22),

$$\deg(I-A, B(\theta, R_1)\backslash(\overline{O_1}\cup\overline{O_4}\cup\overline{B(\theta, C_0)}), \theta) = 1-1+1+1 = 2.$$

This implies that A has at least one fixed point $x_6 \in B(\theta, R_1)\backslash(\overline{O_1}\cup\overline{O_4}\cup\overline{B(\theta, C_0)})$. Obviously, x_5 and x_6 are two distinct sign-changing solutions of the boundary-value problem (1.1). The proof is complete. □

By the method used in the proof of Theorem 2.1, it is easy to show the following four corollaries.

Corollary 3.1. *Suppose that (H0), (H1) and (H3) hold, and that there exists positive integer n_0 such that $\lambda_{2n_0} < \beta_0 < \lambda_{2n_0+1}$. Then the boundary-value problem* (1.1) *has at least one sign-changing solution. Moreover, the boundary-value problem* (1.1) *has at least one positive solution and one negative solution.*

Corollary 3.2. *Suppose that (H0), (H1) and (H3) hold, and that there exists positive integer n_1 such that $\lambda_{2n_1} < \beta_1 < \lambda_{2n_1+1}$. Then the conclusion of Corollary 3.1 holds.*

Corollary 3.3. *Suppose that (H0) and (H1) hold, $\beta_0 > \lambda_1$, $\beta_1 < \lambda_1$ (or $\beta_0 < \lambda_1$, $\beta_1 > \lambda_1$). Then the boundary-value problem* (1.1) *has at least one positive solution and one negative solution.*

Corollary 3.4. *Suppose that (H0), (H1) and (H3) hold, $\beta_0 > \lambda_1$, $\beta_1 > \lambda_1$. Then the boundary-value problem* (1.1) *has at least two positive solutions and two negative solutions.*

Remark. In Theorem 2.1, we show not only the existence of multiple sign-changing solutions, but also the existence of multiple positive solutions and negative solutions. Obviously, we can employ this method to show the existence of sign-changing solutions for other nonlinear boundary-value problems.

Acknowledgment. The author wishes to express his sincere gratitude to the anonymous referees for their helpful suggestions in improving the paper.

References

[1] Guo Dajun, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic press, Inc, New York, 1988.
[2] Guo Dajun, *Nonlinear Anylysis*, Shandong Sci. & Tec. Press, Jinan, 2001.
[3] W. Feng and J. R. L. Webb, *Solvability of a m-point nonlinear boundary-value problem with nonlinear growth*, J. Math. Anal. Appl., 212(1997),467-480.
[4] C. P. Gupta, S. I. Trofimchuk, *Existence of a solution of a three-point boundary-value problem and spectral radius of a related linear operator*, Nonlinear Anal., 34(1998):489-507.
[5] C. P. Gupta, S. I. Trofimchuk, *A sharper condition for the solvability of a three-point second order boundary-value problem*, J. Math. Anal. Appl., 205(1997),586-597.
[6] M. A. Krasnosel'skiĭ, P. P. Zabreĭko, *Geometrical Methods of Nonlinear Analysis*, Springer-Verlag, Berlin Heidelberg, New York, Tokyo, 1984.
[7] X. Liu, *Nontrivial solutions of singular nonlinear m-point boundary-value problems*, J. Math. Anal. Appl.,284(2003),576-590.
[8] Liu Z. L., *On Dancer's conjeture and multiple solutions of elliptic partial differential equations*, Northeast Mathematics J., 9(1993)388-394.

[9] R. Ma, *Existence of solutions of nonlinear m-point boundary-value problems*, J. Math. Anal. Appl., 256, 556-567(2001).

[10] R. Ma, *Positive solutions of a nonlinear three-point boundary-value problem*, Electronic Journal of Differential Equations, Vol. 1998(1998) No. 34, 1-8.

[11] M. Moshinsky, *Sobre los problemas de condiciones a la frontiera en una dimension de caracteristicas discontinuas*, Bol. Soc. Mat. Mexicana 7 (1950) 1C25.

[12] Youssef N. Raffoul, *Positive solutions of three-point nonlinear second order boundary-value problem*, EJQTDE, 2002, Vol. 15, 1-11.

[13] S. Timoshenko, *Theory of Elastic Stability*, McGraw-Hill, New York, 1961.

[14] J. R. L. Webb, *Positive solutions of some three point boundary-value problems via fixed point index*, Nonlinear Analysis, 47(2001), 4319-4332.

[15] X. Xu, *Positive Solutions for Singular m-point Boundary Value Problems with Positive Parameter*, J. Math. Anal. Appl.,291(2004)352-367.

[16] X. Xu, *Multiplicity Results for Positive Solutions of Some Semi-positone Three-point Boundary Value Problems*, J. Math. Anal. Appl., 291(2004)673-689.

[17] Qing Liu Yao, *Existence and multiplicity of positive solutions for a class of second order three-point nonlinear boundary-value problems*, Acta Mathematica Sinica, Vol. 45, No 6, 2002.

Xian Xu

Department of Mathematics, Xuzhou Normal University, Xuzhou, Jiangsu, 221116, China

E-mail address: xuxian68@163.com