Electronic Journal of Differential Equations, Vol. 2004(2004), No. 35, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

EXISTENCE OF SOLUTIONS TO THE ROSENAU AND BENJAMIN-BONA-MAHONY EQUATION IN DOMAINS WITH MOVING BOUNDARY

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Abstract. In this article, we prove the existence of solutions for a hyperbolic equation known as the the Rosenau and Benjamin-Bona-Mahony equations. We study increasing, decreasing, and mixed non-cylindrical domains. Our main tools are the Galerkin method, multiplier techniques, and energy estimates.

1. Introduction

To investigate the dynamics of certain discrete systems, Philip Rosenau obtained the equation $u_t + (u + u^2)_x + u_{xxxxt} = 0$. The study of this equation in cylindrical domains was done by Mi Ai Park [13], who proved the existence and uniqueness of local and global solutions. The Rosenau equation could be seen as a variant of Benjamin-Bona-Mahony (BBM) equation, $u_t + (u + u^2)_x - u_{xxt} = 0$, which models long waves in a non linear dispersive system. In [3], Benjamin-Bona-Mahony proved the existence and uniqueness of global solutions for the BBM equation in cylindrical domains. In this work, we study the existence of solutions for the Rosenau and BBM equations for increasing, decreasing, and mixed noncylindrical domains.

We introduce the following notation: Let α, β, $\gamma = \beta - \alpha$, be C^2-functions of a real variable, such that $\alpha(t) < \beta(t)$, for all $t \geq 0$. We represent the noncylindrical domain by

$$\widehat{Q} = \{(x,t) \in \mathbb{R}^2 : \alpha(t) < x < \beta(t),\ \forall t \geq 0\},$$

and its lateral boundary by $\widehat{\Sigma} = \bigcup_{0 \leq t \leq T} \{\alpha(t), \beta(t)\} \times \{t\}$.

In the present work we investigate the following two equations:

$$\begin{gathered} u_t + (u + u^2)_x + u_{xxxxt} = 0 \quad \text{in } \widehat{Q} \\ u(x,t) = 0 \quad \text{for } (x,t) \in \widehat{\Sigma} \\ u_x(x,t) = 0 \quad \text{for } (x,t) \in \widehat{\Sigma} \\ u(x,0) = u^0(x) \quad \text{for } \alpha(0) < x < \beta(0) \end{gathered} \tag{1.1}$$

1991 *Mathematics Subject Classification.* 35M10, 35B30.
Key words and phrases. Benjamin-Bona-Mahony equation, Rosenau equation, noncylindrical domains.

Submitted April 28, 2003. Published March 11, 2004.

and

$$\begin{gathered} u_t + (u + u^2)_x - u_{xxt} = 0 \quad \text{in } \widehat{Q} \\ u(x,t) = 0 \quad \text{for } (x,t) \in \widehat{\Sigma} \\ u(x,0) = u^0(x) \quad \text{in } \Omega_0 \,. \end{gathered} \tag{1.2}$$

This paper is organized as follows: The next section is devoted to the existence and uniqueness of solution for (1.1) and (1.2), satisfying the hypothesis

(H1) $\alpha'(t) \geq 0$ and $\beta'(t) \leq 0$ for $t \in [0,T]$.

Note that this hypothesis implies $\widehat{Q}$ decreases in the sense that if $t_2 > t_1$, then the projection of $[\alpha(t_2), \beta(t_2)]$ in the subspace $t = 0$ is contained in the projection of $[\alpha(t_1), \beta(t_1)]$ in the same subspace.

In the third section of this article, we study the existence of solutions for (1.1) and (1.2) satisfying the hypothesis

(H2) $\alpha'(t) \leq 0$ and $\beta'(t) \geq 0$ for $t \in [0,T]$.

Analogously hypothesis (H2) implies that $\widehat{Q}$ increases

In the last section of this article, we study the (1.1) and (1.2), satisfying the hypothesis:

(H3) $\widehat{Q} = \widehat{Q_1} \cup \widehat{Q_2}$ where $\widehat{Q_1}$ is increasing and $\widehat{Q_2}$ is decreasing.

In the following, by Ω we represent the interval $]0,1[$, Ω_t and Ω_0 denote the intervals $]\alpha(t), \beta(t)[$ and $]\alpha(0), \beta(0)[$ respectively. We denote, as usual, by $(.,.)$, $\|\cdot\|$ respectively the scalar product and norm in $L^2(\Omega)$. In the sequel, $w_{m,x}$ denotes $\frac{\partial w_m}{\partial x}$, analogously $w_{m,xx} = \frac{\partial^2 w_m}{\partial x^2}$, $w_{m,xt} = \frac{\partial^2 w_m}{\partial t\, \partial x}$, etc.

2. Solutions on decreasing domains

In this section we study the existence and uniqueness for (1.1) and (1.2) satisfying the hypothesis (H1). Let $\gamma(t) = \beta(t) - \alpha(t) > 0$, for all $t \geq 0$. Then $0 < \frac{x-\alpha(t)}{\gamma(t)} < 1$, for all $t \in [0,T]$. With the change of variable $u(x,t) = v(y,t)$ where $y = \frac{x-\alpha(t)}{\gamma(t)}$, for all $t \in [0,T]$, problem (1.1) is transformed into

$$\begin{gathered} v_t + \frac{1}{\gamma}(v + v^2)_y + \frac{1}{\gamma^4} v_{yyyyt} - \frac{(\alpha' + \gamma' y)}{\gamma} v_y - \frac{4\gamma'}{\gamma^5} v_{yyyy} \\ - \frac{(\alpha' + \gamma' y)}{\gamma^5} v_{yyyyy} = 0 \quad \text{in } \Omega \times]0,T[\\ v(0,t) = v(1,t) = 0 \quad \text{in }]0,T[\\ v_y(0,t) = v_y(1,t) = 0 \quad \text{in }]0,T[\\ v(y,0) = v^0(y) \quad \text{in } \Omega \,. \end{gathered} \tag{2.1}$$

Also problem (1.2) is transformed into

$$\begin{gathered} v_t + \frac{1}{\gamma}(v + v^2)_y - \frac{1}{\gamma 2} v_{yyt} - \frac{(\alpha' + \gamma' y)}{\gamma} v_y + \frac{2\gamma'}{\gamma^3} v_{yy} \\ + \frac{(\alpha' + \gamma' y)}{\gamma^3} v_{yyy} = 0 \quad \text{in } \Omega \times]0,T[\\ v(0,t) = v(1,t) = 0 \quad \text{in }]0,T[\\ v(y,0) = v^0(y) \quad \text{in } \Omega \,. \end{gathered} \tag{2.2}$$

Under these conditions, we establish the following existence results.

Theorem 2.1. *For each $u^0 \in H_0^2(\Omega_0) \cap H^4(\Omega_0)$, there exists a unique function $u : \widehat{Q} \to \mathbb{R}$, satisfying $u \in C^1([0,T]; H_0^2(\Omega_t)) \cap C(0,T; H^3(\Omega_t) \cap H_0^2(\Omega_t))$ and*

$$\int_{\widehat{Q}} u_t \phi \, dx \, dt + \int_{\widehat{Q}} (u+u^2)_x \phi \, dx \, dt + \int_{\widehat{Q}} u_{xxt} \phi_{xx} \, dx \, dt = 0,$$

for all $\phi \in L^2(0,T; H_0^2(\Omega_t))$, $u(x,0) = u^0(x)$, for all $x \in \Omega_0$.

Theorem 2.2. *For each $u^0 \in H_0^1(\Omega_0) \cap H^2(\Omega_0)$, there exists a unique function $u : \widehat{Q} \to \mathbb{R}$, satisfying $u \in L^\infty(0,T; H_0^1(\Omega_t))$, $u_t \in L^\infty(0,T; H_0^1(\Omega_t))$ and*

$$\int_{\widehat{Q}} u_t \phi \, dx \, dt + \int_{\widehat{Q}} (u+u^2)_x \phi \, dx \, dt + \int_{\widehat{Q}} u_{xt} \phi_x \, dx \, dt = 0,$$

for all $\phi \in L^2(0,T; H_0^1(\Omega_t))$, $u(x,0) = u^0(x)$, for all $x \in \Omega_0$.

To prove Theorem 2.1, we need the following lemmas.

Lemma 2.3. *For each $v^0 \in H_0^2(\Omega) \cap H^4(\Omega)$, there exists a unique function $v : \Omega \times]0,T[\to \mathbb{R}$, satisfying $v \in L^\infty(0,T; H_0^2(\Omega) \cap H^4(\Omega))$, $v_t \in L^\infty(0,T; H_0^2(\Omega))$, and*

$$\begin{aligned} &\int_{\Omega\times]0,T[} [v_t\psi + \frac{1}{\gamma}(v+v^2)_y \psi + \frac{1}{\gamma^4} v_{yyt}\psi_{yy} \\ &-\frac{(\alpha'+\gamma' y)}{\gamma} v_y \psi - \frac{4\gamma'}{\gamma^5} v_{yy}\psi_{yy} + (\frac{(\alpha'+\gamma' y)}{\gamma^5}\psi)_y v_{yyyy}] dy\, dt = 0, \end{aligned}$$

for all $\psi \in L^2(0,T; H_0^2(\Omega))$, $v(y,0) = v^0(y)$, for all $y \in \Omega$.

Lemma 2.4. *For each $f \in C([0,T]; H^{-2}(\Omega))$, there exists a unique function $z : \Omega \times]0,T[\to \mathbb{R}$, satisfying $z \in C([0,T]; H_0^2(\Omega))$ and $z + \frac{1}{\gamma^4}\Delta^2 z = f$*

Lemma 2.5. *For each $v^0 \in H_0^2(\Omega) \cap H^4(\Omega)$, there exists a unique function $v : \Omega \times]0,T[\to \mathbb{R}$, satisfying $v \in C^1([0,T]; H_0^2(\Omega)) \cap C([0,T]; H^3(\Omega) \cap H_0^2(\Omega))$ and*

$$\begin{aligned} &\int_{\Omega\times]0,T[} [v_t\psi + \frac{1}{\gamma}(v+v^2)_y \psi + \frac{1}{\gamma^4} v_{yyt}\psi_{yy} \\ &-\frac{(\alpha'+\gamma' y)}{\gamma} v_y \psi - \frac{4\gamma'}{\gamma^5} v_{yy}\psi_{yy} + (\frac{(\alpha'+\gamma' y)\psi}{\gamma^5})_y v_{yyyy}\,] \, dy\, dt = 0\,, \end{aligned}$$

for all $\psi \in L^2(0,T; H_0^2(\Omega))$; $v(y,0) = v^0(y)$, for all $y \in \Omega$.

Lemma 2.6. *For each $v^0 \in H_0^1(\Omega) \cap H^2(\Omega)$, there exists a unique function $v : \Omega \times]0,T[\to \mathbb{R}$, satisfying $v \in L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega))$, $v_t \in L^\infty(0,T; H_0^1(\Omega))$ and*

$$\begin{aligned} &\int_{\Omega\times]0,T[} [v_t\psi + \frac{1}{\gamma}(v+v^2)_y \psi - \frac{1}{\gamma^2} v_{yyt}\psi \\ &-\frac{(\alpha'+\gamma' y)}{\gamma} v_y \psi + \frac{2\gamma'}{\gamma^3} v_{yy}\psi - (\frac{(\alpha'+\gamma' y)\psi}{\gamma^3})_y v_{yy}\,] dy\, dt = 0, \end{aligned}$$

for all $\psi \in L^2(0,T; H_0^1(\Omega))$; $v(y,0) = v^0(y)$, for all $y \in \Omega$.

In this article, we prove Theorem 2.1 and Lemmas 2.3, 2.4, 2.5 which correspond to Rosenau Equation. However, we omit the proofs of Theorem 2.2 and Lemma 2.6 which correspond to Benjamin Bona-Mahony Equation; because the proofs are made in a similar way.

Proof of Lemma 2.3. Let $(w_i)_{i\in\mathbb{N}}$ be the special basis of $H_0^2(\Omega)$, such that

$$w_{i,yyyy} = \lambda_i w_i, \quad \text{in } \Omega$$
$$w_i(0) = w_i(1) = w_{i,y}(0) = w_{i,y}(1) = 0, \quad i \in \mathbb{N}.$$

We denote by V_m the subspace generated by $w_1, \dots, w_m$. Our starting point is to construct the Galerkin approximation of the solution $v_m \in V_m$, which is given by the solution of the approximate equation

$$\begin{gathered}(v_{m,t}, w) + (\frac{1}{\gamma}(v_m + v_m^2)_y, w) + \frac{1}{\gamma^4}(v_{m,yyyyt}, w) - \frac{4\gamma'}{\gamma^5}(v_{m,yyyy}, w) \\ +(-\frac{(\alpha' + \gamma' y)}{\gamma^5} v_{m,yyyyy}, w) + (-\frac{(\alpha' + \gamma' y)}{\gamma} v_{m,y}, w) = 0 \quad \text{for all } w \in V_m \\ v_m(0) = v_m^0 \to v^0 \quad \text{in } H^4(\Omega)\end{gathered} \tag{2.3}$$

First Estimate. Taking $w = v_m(t)$ in $(\mathrm{V})_1$, we have:

$$\begin{aligned}&\frac{d}{dt}(\|v_m(t)\|^2 + \frac{1}{\gamma^4}\|v_{m,yy}(t)\|^2) + \frac{\gamma'}{\gamma^5}\|v_{m,yy}(t)\|^2 \\ &+\frac{\gamma'}{\gamma}\|v_m(t)\|^2 + \frac{\alpha'}{\gamma^5}v_{m,yy}^2(0) - \frac{\beta'}{\gamma^5}v_{m,yy}^2(1) = 0\end{aligned} \tag{2.4}$$

Integrating this equation over $[0,t]$ and using hypothesis (H1), we obtain

$$\begin{aligned}&\|v_m(t)\|^2 + \frac{1}{\gamma^4}\|v_{m,yy}(t)\|^2 \\ &\le \|v^0\|^2 + \frac{1}{\gamma^4(0)}\|v_{yy}^0\|^2 + \frac{\gamma_2}{\gamma_0}\int_0^t \frac{1}{\gamma^4}\|v_{m,yy}(s)\|^2 ds + \frac{\gamma_2}{\gamma_0^5}\int_0^t \|v_m(s)\|^2 ds\end{aligned} \tag{2.5}$$

where γ_0, γ_1, γ_2 are positive constants, such that $\gamma_0 \le \gamma(t) \le \gamma_1$, and

$$\gamma_2 = \max_{0\le t\le T}|\gamma'(t)|, \quad \gamma_1 = \max_{0\le t\le T}|\alpha'(t)|.$$

This implies

$$\|v_m(t)\|^2 + \frac{1}{\gamma^4}\|v_{m,yy}(t)\|^2 \le C_0 + C_1\int_0^t [\|v_m(s)\|^2 + \frac{1}{\gamma^4}\|v_{m,yy}(s)\|^2]\, ds \tag{2.6}$$

where $C_0, C_1, \dots$ denote positive constants. Applying Gronwall inequality, we have the first estimate

$$\|v_m(t)\|^2 + \frac{1}{\gamma^4}\|v_{m,yy}(t)\|^2 \le C_2 \tag{2.7}$$

Second Estimate Taking $w = v_{m,yyyy}(t)$ in the first equation of (2.3), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}[\|v_{m,yy}(t)\|^2 + \frac{1}{\gamma^4}\|v_{m,yyyy}(t)\|^2] \\ &\le \frac{3\gamma_2}{2\gamma_0}\frac{1}{\gamma^4}\|v_{m,yyyy}(t)\|^2 + \frac{1}{\gamma}\|v_{m,y}(t)\|\|v_{m,yyyy}(t)\| \\ &\quad + \frac{(\alpha_1 + \gamma_2)}{\gamma_0}\|v_{m,y}(t)\|^2\|v_{m,yyyy}(t)\|.\end{aligned} \tag{2.8}$$

From (2.7) and using Schwartz's inequality and Poincare's inequality, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[\|v_{m,yy}(t)\|^2+\frac{1}{\gamma^4}\|v_{m,yyyy}(t)\|^2]\\
&\le \frac{3\gamma_2}{2\gamma_0}\frac{1}{\gamma^4}\|v_{m,yyyy}(t)\|^2+\frac{1}{2\gamma^4}\|v_{m,yyyy}(t)\|^2+\frac{\gamma_1^2}{2}C_3\|v_{m,yy}(t)\|^2\\
&\quad +C_3\|v_{m,yy}(t)\|^2\|v_{m,yyyy}(t)\|\\
&\le \frac{1}{2}(1+\frac{3\gamma_2}{\gamma_0})\frac{1}{\gamma^4}\|v_{m,yyyy}(t)\|^2+\frac{\gamma_1^2}{2}C_3C_2+\frac{(\alpha_1+\gamma_2)}{\gamma_0}C_3C_2\|v_{m,yyyy}(t)\|\\
&\le \frac{1}{2}(1+\frac{3\gamma_2}{\gamma_0})\frac{1}{\gamma^4}\|v_{m,yyyy}(t)\|^2+\frac{\gamma_1^2}{2}C_3C_2+\frac{\gamma_1^2}{2}C_4^2+\frac{1}{2\gamma^4}\|v_{m,yyyy}(t)\|^2\,.
\end{aligned}
$$

Let $c_4=\frac{(\alpha_1+\gamma_2)}{\gamma_0}C_3C_2$ and let $c_5=\frac{\gamma_1^2}{2}C_3C_2+\frac{\gamma_1^2}{2}C_4^2$. Then

$$
\frac{d}{dt}[\|v_{m,yy}(t)\|^2+\frac{1}{\gamma^4}\|v_{m,yyyy}(t)\|^2]\le C_5+(2+\frac{3\gamma_2}{\gamma_0})\frac{1}{\gamma^4}\|v_{m,yyyy}(t)\|^2. \tag{2.9}
$$

Integrating this inequality over $[0,t]$ and applying Gronwall inequality, we obtain

$$
\|v_{m,yy}(t)\|^2+\frac{1}{\gamma^4}\|v_{m,yyyy}(t)\|^2\le C_6 \tag{2.10}
$$

Third Estimate Taking $w=v_{m,t}(t)$ in $(\mathrm{V})_1$, we have

$$
\begin{aligned}
&\|v_{m,t}(t)\|^2+\frac{1}{\gamma^4}\|v_{m,yyt}(t)\|^2\\
&=-\frac{1}{\gamma}((v_m(t)+v_m^2(t))_y,v_{m,t}(t))+\frac{4\gamma'}{\gamma^5}(v_{m,yyyy}(t),v_{m,t}(t))\\
&\quad -(\frac{(\alpha'+\gamma'y)}{\gamma^5}v_{m,yyyy}(t),v_{m,yt}(t))+(\frac{(\alpha'+\gamma'y)}{\gamma^5}v_{m,y}(t),v_{m,t}(t)).
\end{aligned} \tag{2.11}
$$

¿From (2.7), (2.10), and (2.11), we obtain

$$
\|v_{m,t}(t)\|^2+\frac{1}{\gamma^4}\|v_{m,yyt}(t)\|^2\le C_7. \tag{2.12}
$$

These three estimates permit to pass to the limit in the approximate equation and we obtain a weak solution v in the sense of Lemma 2.3. The uniqueness of solution and the verification of initial data are showed by the standard arguments. □

Proof of Lemma 2.4. To prove the existence we consider two stages: First stage $f\in C([0,T];H_0^2(\Omega))$. Let $(w_i)_{i\in\mathbb{N}}$ be the special basis of $H_0^2(\Omega)$ used in the proof of Lemma 2.3. Consider the sequence (f_n), such that $f_n(t)=\sum_{i=1}^n(f(t),w_i)w_i$. It is clear that $f_n\to f$ strongly in $C([0,T];H_0^2(\Omega))$.

The approximated solution $z_m(t)$ to $z+\frac{1}{\gamma^4}\Delta^2z=f$ is $z_m(t)=\sum_{i=1}^m g_{im}(t)w_i$, where g_{im} are solutions of the approximated system

$$
(z_m(t),w_i)+\frac{1}{\gamma^4}(\Delta z_m(t),\Delta w_i)=(f_m(t),w_i)\,,\quad i=1,\dots m \tag{2.13}
$$

A priori estimate. Let us prove that (z_m) is a Cauchy sequence in $C([0,T];H_0^2(\Omega))$. In fact, let m and n be positive integer such that $m>n$ and $g_{in}(t)=0$ for

$n \le i \le m$. Then z_m and z_n are solutions of (2.13) in V_m. Consider the Cauchy difference $z_m - z_n$. We have from (2.13) that, for $i = 1, \dots, m$,

$$(z_m(t) - z_n(t), w_i) + \frac{1}{\gamma^4}(\Delta(z_m(t) - z_n(t)), \Delta w_i) = (f_m(t) - f_n(t), w_i)\,, \tag{2.14}$$

Taking $w_i = z_m(t) - z_n(t)$ in (2.14), using the Cauchy-Schwarz inequality and the equivalent norms, we obtain

$$|z_m - z_n|_{C([0,T];H_0^2(\Omega))} \le c|f_m - f_n|_{C([0,T];H_0^2(\Omega))}$$

Then $z_n \to z$ strongly in $C([0,T]; H_0^2(\Omega))$. Therefore, taking limit in (2.13), we obtain $z + \frac{1}{\gamma^4}\Delta^2 z = f$ in $C([0,T]; H^{-2}(\Omega))$.

Second stage $f \in C([0,T]; H^{-2}(\Omega))$. By density, there exists a sequence (f_n), $f_n \in C([0,T]; H_0^2(\Omega))$, such that $f_n \to f$ strongly in $C([0,T]; H^{-2}(\Omega))$. Using the first stage we have that there exist a sequence (z_n), $z_n \in C([0,T]; H_0^2(\Omega))$ such that

$$z_n + \frac{1}{\gamma^4}\Delta^2 z_n = f_n \quad \text{in } C([0,T]; H^{-2}(\Omega)). \tag{2.15}$$

Consider the Cauchy difference $z_m - z_n$, $m > n$. We obtain

$$z_{m-} z_n + \frac{1}{\gamma^4}\Delta^2(z_m - z_n) = f_m - f_n \quad \text{in } C([0,T]; H^{-2}(\Omega)). \tag{2.16}$$

Composing (2.16) with $z_m - z_n \in C([0,T]; H_0^2(\Omega))$. and integrating in Ω, we have

$$|z_m - z_n|_{C([0,T];H_0^2(\Omega))} \le c|f_m - f_n|_{C([0,T];H^{-2}(\Omega))};$$

therefore, $z_n \to z$ strongly in $C([0,T]; H_0^2(\Omega))$ and taking limit in (2.15) we have $z + \frac{1}{\gamma^4}\Delta^2 z = f$ in $C([0,T]; H^{-2}(\Omega))$. The uniqueness of the solutions is showed by the standard arguments. □

Proof of Lemma 2.5. From Lemma 2.4 we can define the operator $\mathcal{B}(t) = (I + \frac{1}{\gamma^4}\Delta^2)^{-1}$ from $C([0,T]; H^{-2}(\Omega))$ to $C([0,T]; H_0^2(\Omega))$ by $\mathcal{B}(t)f = z$ with $f \in C([0,T]; H^{-2}(\Omega))$ where z is a solution of $z + \frac{1}{\gamma^4}\Delta^2 z = f$. Note that $\mathcal{B}(t)$ is linear and continuous.

By Lemma 2.3, $v \in L^2(0,T; H^4(\Omega) \cap H_0^2(\Omega))$ and $v_t \in L^2(0,T; H_0^2(\Omega))$. From Lions-Magenes, Theorems 3.1 and 9.6, chapter I [10], we conclude that

$$v \in C([0,T]; H_0^2(\Omega) \cap H^3(\Omega)) \tag{2.17}$$

On the other hand, from the transformed problem (2.1), we obtain

$$(I + \frac{1}{\gamma^4}\Delta^2)v_t = f\,, \tag{2.18}$$

where,

$$f = -\frac{1}{\gamma}(v + v^2)_y + \frac{(\alpha' + \gamma' y)}{\gamma}v_y + \frac{4\gamma'}{\gamma^5}v_{yyyy} + (\frac{(\alpha' + \gamma' y)}{\gamma^5})v_{yyyyy}$$

From (2.17), we conclude that $f \in C([0,T]; H^{-2}(\Omega))$. Then from (2.18) we have that $v_t = \mathcal{B}(t)f$, where $v_t \in C([0,T]; H_0^2(\Omega))$ and we get the required result. □

The proof of Theorem 2.1 follows immediately from Lemma 2.5 and the Change of Variable Theorem. Therefore, we omit it.

Observe that Theorem 2.1 in a cylindrical domain, has the regularity $u \in C^1([0,T]; H_0^2(\Omega)) \cap C([0,T]; H^4(\Omega) \cap H_0^2(\Omega))$. In fact, as we consider an additional estimate with $w_i = u_{m,txxxx}$, in the Galerkin approximation, that allows us to obtain the regularity $u_t \in L^2(0,T;H^4(\Omega)\cap H_0^2(\Omega))$. However, in our noncylindrical domain, this is not possible since the transformed problem (III), contains a term v_{yyyyy} that does not allow us to use the estimate with $w_i = v_{m,tyyyy}$ in the Galerkin approximation.

3. Solutions on increasing domains

In this section we study the existence of solution for the systems (1.1) and (1.2) satisfying the hypothesis (H2). We use the Penalization Method given by Lions [10]. Let $Q =]a,b[\times]0,T[$ be the cylinder such that $\widehat{Q}\subset Q$. We define the function $M: Q\to\mathbb{R}$, by

$$M(x,t)=\begin{cases}1 & \text{in } Q\setminus\widehat{Q}\\ 0 & \text{in } \widehat{Q}\end{cases}$$

To show the existence result we will use the following Lemma.

Lemma 3.1. *If $u, u_t \in L^2(0,T;L^2(a,b))$, then*

$$\int_0^t (Mu(s),u_t(s))ds \geq \frac{1}{2}\|M(t)u(t)\|^2_{L^2(a,b)} - \frac{1}{2}\|M(0)u(0)\|^2_{L^2(a,b)}\,.$$

Proof. We have

$$\begin{aligned}\int_0^t (Mu(s),u_t(s))ds &= \frac{1}{2}\int_0^t\int_a^b M(u^2(s))_t\,d\xi\,ds\\ &= \frac{1}{2}\int_{[0,t]\times[a,b]} M(u^2(s))_t\,d\xi\,ds.\end{aligned}$$

From Fubini's Theorem and recalling the definition of M, it follows that

$$\begin{aligned}&\int_0^t (Mu(s),u_t(s))ds\\
&= \frac{1}{2}\int_a^{\alpha(t)}\int_0^t [u^2(s)]_t\,ds\,d\xi + \frac{1}{2}\int_{\alpha(t)}^{\alpha(0)}\int_0^{\alpha^{-1}(x)}[u^2(s)]_t\,ds\,d\xi\\
&\quad + \frac{1}{2}\int_{\beta(0)}^{\beta(t)}\int_0^{\beta^{-1}(x)}[u^2(s)]_t\,ds\,d\xi + \frac{1}{2}\int_{\beta(t)}^b\int_0^t [u^2(s)]_t\,ds\,d\xi\\
&= \frac{1}{2}\int_a^{\alpha(t)}[u^2(t,\xi)-u^2(0,\xi)]\,d\xi + \frac{1}{2}\int_{\alpha(t)}^{\alpha(0)}[u^2(\alpha^{-1}(x),0)-u^2(0,\xi)]\,d\xi\\
&\quad + \frac{1}{2}\int_{\beta(0)}^{\beta(t)}[u^2(\beta^{-1}(x),0)-u^2(0,\xi)]\,d\xi + \frac{1}{2}\int_{\beta(t)}^b[u^2(t,\xi)-u^2(0,\xi)]\,d\xi\\
&\geq \frac{1}{2}[\int_a^{\alpha(t)}u^2(t,\xi)\,d\xi + \int_{\beta(t)}^b u^2(t,\xi)\,d\xi - (\int_a^{\alpha(t)}u^2(0,\xi)\,d\xi\\
&\quad + \int_{\alpha(t)}^{\alpha(0)}u^2(0,\xi)\,d\xi + \int_{\beta(0)}^{\beta(t)}u^2(0,\xi)\,d\xi + \int_{\beta(t)}^b u^2(0,\xi)\,d\xi)]\\
&= \frac{1}{2}\int_a^b M(t,\xi)\,u^2(t,\xi)\,d\xi - \frac{1}{2}\int_a^b M(0,\xi)u^2(0,\xi)\,d\xi\\
&= \frac{1}{2}\|M(t)u(t)\|^2_{L^2(a,b)} - \frac{1}{2}\|M(0)u(0)\|^2_{L^2(a,b)}\end{aligned}$$

which completes the proof. □

The existence of solution for (1.1) and (1.2), satisfying the hypothesis (H2), is established in the next theorems.

Theorem 3.2. *For each $u^0 \in H_0^2(\Omega_0)$, there exists a function $u : \widehat{Q} \to \mathbb{R}$, satisfying $u \in L^\infty(0,T;H_0^2(\Omega_t))$, $u_t \in L^\infty(0,T;H_0^2(\Omega_t))$ and*

$$\int_{\widehat{Q}} u_t \phi \, dx\, dt + \int_{\widehat{Q}} (u+u^2)_x \phi \, dx\, dt + \int_{\widehat{Q}} u_{xxt}\phi_{xx}\, dx\, dt = 0 \tag{3.1}$$

for all $\phi \in L^2(0,T;H_0^2(\Omega_t))$; $u(x,0) = u^0(x)$

Theorem 3.3. *For each $u^0 \in H_0^1(\Omega_0)$, there exists a function $u : \widehat{Q} \to \mathbb{R}$, satisfying $u \in L^\infty(0,T;H_0^1(\Omega_t))$, $u_t \in L^\infty(0,T;H_0^1(\Omega_t))$ and*

$$\int_{\widehat{Q}} u_t \phi \, dx\, dt + \int_{\widehat{Q}} (u+u^2)_x \phi \, dx\, dt + \int_{\widehat{Q}} u_{xt}\phi_{x}\, dx\, dt = 0,$$

for all $\phi \in L^2(0,T;H_0^1(\Omega_t))$; $u(x,0) = u^0(x)$

Proof of Theorem 3.2. To prove this result we use the penalization method. For each $\epsilon > 0$ we consider the problem

$$\begin{gathered} u_{\epsilon,t} + (u_\epsilon + u_\epsilon^2)_x + u_{\epsilon,xxxxt} + \frac{1}{\epsilon} M u_{\epsilon,t} - \frac{1}{\epsilon}(M u_{\epsilon,xt})_x = 0 \quad \text{in } Q \\ u_\epsilon(a,t) = u_\epsilon(b,t) = u_{\epsilon,x}(a,t) = u_{\epsilon,x}(b,t) = 0 \quad \text{in }]0,T[\\ u_\epsilon(x,0) = \widetilde{u}^0(x) \quad \text{in }]a,b[\end{gathered} \tag{3.2}$$

Let $\{w_i\}_{i\in N}$ be a basis of $H_0^2(a,b)$, such that $w_1 = \widetilde{u}_0$. We denote by $V_m = [w_1,\dots,w_m]$ the subspace of $H_0^2(a,b)$, generated by $\widetilde{u}_0, w_2,\dots,w_m$. We seek $u_{\epsilon m}(t)$ in V_m solution to the approximate problem

$$\begin{gathered} (u_{\epsilon m,t}, w) + ((u_{\epsilon m} + u_{\epsilon m}^2)_x, w) + (u_{\epsilon m,xxxxt}, w) \\ + \frac{1}{\epsilon}(M u_{\epsilon m,t}, w) - \frac{1}{\epsilon}((M u_{\epsilon m,xt})_x, w) = 0 \quad \text{for all } w \in V_m \\ u_{\epsilon m}(0) = u^0(x) \to \widetilde{u}^0 \quad \text{in } H_0^2(a,b) \end{gathered} \tag{3.3}$$

First Estimate. Taking $w = u_{\epsilon m}$ in (3.3) and applying Lemma 3.1, we obtain

$$|||u_{\epsilon m}(t)|||^2 + |||u_{\epsilon m,xx}(t)|||^2 + \frac{1}{\epsilon}|||M(t)u_{\epsilon m}(t)|||^2 + \frac{1}{\epsilon}|||M(t)u_{\epsilon m,x}(t)|||^2 \le c_8, \tag{3.4}$$

where $|||\cdot|||$ denotes the norm in $L^2(a,b)$.

Second Estimate. Taking $w = u_{\epsilon m,t}(t)$ in (3.3) and using (3.4) we have

$$\begin{aligned} &|||u_{\epsilon m,t}(t)|||^2 + |||u_{\epsilon m,xxt}(t)|||^2 + \frac{1}{\epsilon}(M(t)u_{\epsilon m,t}(t), u_{\epsilon m,t}(t)) \\ &\frac{1}{\epsilon}(M(t)u_{\epsilon m,xt}(t), u_{\epsilon m,xt}(t)) \\ &\le c_9 + \frac{1}{2}|||u_{\epsilon m,t}(t)|||^2 \end{aligned} \tag{3.5}$$

From where we obtain

$$|||u_{\epsilon m,t}(s)|||^2 + |||u_{\epsilon m,xxt}(s)|||^2 + \frac{1}{\epsilon}|||M(t)u_{\epsilon m,t}(t)|||^2 + \frac{1}{\epsilon}|||M(t)u_{\epsilon m,xt}(t)|||^2 \le c_9$$

From the estimates above, we pass to the limit in the approximate equation, and we obtain that u_ϵ is solution of the penalized problem

$$\begin{aligned}\int_0^T\int_a^b u_{\epsilon,t}v\,dx\,dt+\int_0^T\int_a^b (u_\epsilon+u_\epsilon^2)_x v\,dx\,dt+\int_0^T\int_a^b u_{\epsilon,xxt}v_{xx}\,dx\,dt\\ +\frac{1}{\epsilon}\int_0^T\int_a^b Mu_{\epsilon,t}v\,dx\,dt+\frac{1}{\epsilon}\int_0^T\int_a^b Mu_{\epsilon,xt}v_x\,dx\,dt=0\end{aligned}\tag{3.6}$$

for all $v\in L^2(0,T;H_0^2(a,b))$. From (3.4), (3.5) and the Banach-Steinhauss Theorem, we pass to the limit as $\epsilon\to 0$ in (3.6) and we obtain (3.1).
Regularity. From the first estimate, we have

$$\frac{1}{\epsilon}\int_0^t (Mu_{\epsilon m,t}(s),u_{\epsilon m}(s))\,ds\le c$$

On the other hand, from Lemma 3.1 we obtain

$$\frac{1}{\epsilon}\int_0^t (Mu_{\epsilon m,t}(s),u_{\epsilon m}(s))\,ds\ge\frac{1}{2\epsilon}|||M(t)u_{\epsilon m}(t)|||^2.$$

Then $|||M(t)u_{\epsilon m}(t)|||^2\le 2c\epsilon$. Thus $\int_0^T\int_a^b M(t)\,u_{\epsilon m}^2(t)\,dx\,dt\le 2c\epsilon T$ or

$$\int_0^T\int_a^b |M(t)u_\epsilon(t)|^2\,dx\,dt\le\liminf\int_0^T\int_a^b |M(t)u_\epsilon(t)|^2\,dx\,dt\le 2c\epsilon T$$

Then $Mu_\epsilon\to 0$ in $L^2(0,T;L^2(a,b))$.

On the other hand, $Mu_\epsilon\to Mu$ in $L^2(0,T;L^2(a,b))$ and $Mu_{\epsilon m}\to Mu_\epsilon$ in $L^2(0,T;L^2(a,b))$. So, we conclude that $Mu=0$ a.e. in Q or $u=0$ in $Q\setminus\widehat{Q}$.

Analogously, applying the Lemma 3.1 to $u_{\epsilon m,xt}$ instead of $u_{\epsilon m,t}$, we obtain: $u_x=0$ in $Q\setminus\widehat{Q}$. Since $u\in L^\infty(0,T;H_0^1(a,b))$ and $u_t\in L^\infty(0,T;H_0^1(a,b))$, then $u\in C([0,T];H_0^1(a,b))$. Therefore, $u(t)\in H_0^1(a,b)$ for all t and $u=0$ in $]a,b[\setminus]\alpha(t),\beta(t)[$. From where $u(t)\in H_0^1(\alpha(t),\beta(t))$, for all t. Thus $u\in L^\infty(0,T;H_0^1(\Omega_t))$. Analogously, $u_x\in L^\infty(0,T;H_0^1(\Omega_t))$. From these two statements, we have that $u\in L^\infty(0,T;H_0^2(\Omega_t))$. From the second estimate,

$$\int_0^T\int_a^b |M(t)u_{\epsilon m,t}(t)|^2\,dx\,dt+\int_0^T\int_a^b |M(t)u_{\epsilon m,xt}(t)|^2\,dx\,dt\le 2c\epsilon T.$$

By similar arguments, we obtain that $u_t\in L^\infty(0,T;H_0^2(\Omega_t))$, which prove the regularity of the solution. □

The proof of Theorem 3.3 is similar to the proof of Theorem 3.2 and is ommitted.

Remark 3.4. Theorems 2.1, 2.2, 3.2 and 3.3 are invariable by translation. In fact, the particular problem

$$\begin{gathered}u_t+(u+u^2)_x+u_{xxxxt}=0\quad\text{in }\widehat{Q}\subset\Omega\times]T_0,T_1[\\ u(x,t)=0\quad\text{in }\widehat{\Sigma}\\ u_x(x,t)=0\quad\text{in }\widehat{\Sigma}\\ u(x,T_0)=u^0(x)\quad\text{in }\Omega_{T_0},\end{gathered}$$

with the change of variable $u(x,t)=\overline{u}(x,t-T_0)$, can be transformed into a problem of type (1.1).

4. Solutions on mixed domains

Here we analyze the case when $\widehat{Q}$ is a mixed domain; i.e., $\widehat{Q} = B_1 \cup B_2$ where $B_1 = \{(x,t) \in \widehat{Q} : 0 < t \le T_1\}$ and $B_2 = \{(x,t) \in \widehat{Q} : T_1 < t < T\}$, where B_1 is decreasing satisfying (H1), and B_2 is increasing satisfying (H2). We define $\widehat{Q_i}$ by $\widehat{Q_i} = \operatorname{int}(B_i)$, $i = 1, 2$. i.e.,

$$\widehat{Q_1} = \{(x,t) \in \widehat{Q} : 0 < t < T_1\} \quad \text{and} \quad \widehat{Q_2} = \{(x,t) \in \widehat{Q}; \quad T_1 < t < T\}.$$

To find a solution to (1.1) in $\widehat{Q} = \widehat{Q_1} \cup \widehat{Q_2}$, we consider the following two cases:
(1) Solution on $\widehat{Q_1}$: For each $u^0 \in H_0^2(\Omega_0) \cap H^4(\Omega_0)$, by Theorem 2.1, there exist u_1 solution of

$$\begin{gathered} u_{1,t} + (u_1 + u_1^2)_x + u_{1,xxxxt} = 0 \quad \text{in } \widehat{Q}_1 \\ u_1(x,t) = 0 \quad \text{in} \widehat{\Sigma}_1 \\ u_{1,x}(x,t) = 0 \quad \text{in} \widehat{\Sigma}_1 \\ u_1(x,0) = u^o(x) \quad \text{in } \Omega_0 \end{gathered} \tag{4.1}$$

satisfying $u_1 \in L^\infty(0, T_1; H_0^2(\Omega_t))$, $u_{1t} \in L^\infty(0, T_1; H_0^2(\Omega_t))$; therefore u_1 is in $C([0, T_1]; H_0^2(\Omega_t))$.
(2) Solution on $\widehat{Q_2}$: For each $\overline{u}^0 = u_1(T_1) \in H_0^2(\Omega_{T_1})$, by Theorem 3.2, there exist u_2 solution of

$$\begin{gathered} u_{2,t} + (u_2 + u_2^2)_x + u_{2,xxxxt} = 0 \quad \text{in } \widehat{Q}_2 \\ u_2(x,t) = 0 \quad \text{in} \widehat{\Sigma}_2 \\ u_{2,x}(x,t) = 0 \quad \text{in} \widehat{\Sigma}_2 \\ u_2(x,T_1) = \overline{u}^0(x) \quad \text{in } \Omega_{T_1} \end{gathered} \tag{4.2}$$

satisfying u_2 in $L^\infty(T_1, T; H_0^2(\Omega_t))$, u_{2t} in $L^\infty(T_1, T; H_0^2(\Omega_t))$, and u_2 in $C([T_1, T]; H_0^2(\Omega_t))$.
(3) Solution on $\widehat{Q}$: We define $u : \widehat{Q} \to \mathbb{R}$ by

$$u(x,t) = \begin{cases} u_1(x,t), & (x,t) \in \widehat{Q_1} \\ u_2(x,t), & (x,t) \in \widehat{Q_2}, \end{cases}$$

where u_1 and u_2 are solutions of (4.1) and (4.2), respectively. Given that $u^0 \in H_0^2(\Omega_0)$, by Remark 3.4, we deduce that the function u defined above is solution of

$$\begin{gathered} u_t + (u + u^2)_x + u_{xxxxt} = 0 \quad \text{in } \widehat{Q} \subset \Omega \times]0, T[\\ u(x,t) = 0 \quad \text{in } \widehat{\Sigma} \\ u_x(x,t) = 0 \quad \text{in } \widehat{\Sigma} \\ u(x,0) = u^0(x) \quad \text{in } \Omega_0 \end{gathered} \tag{4.3}$$

satisfying $u \in L^\infty(0, T; H_0^2(\Omega_t))$, $u_t \in L^\infty(0, T; H_0^2(\Omega_t))$ and

$$\int_{\widehat{Q}} u_t \phi \, dx \, dt + \int_{\widehat{Q}} (u + u^2)_x \phi \, dx \, dt + \int_{\widehat{Q}} u_{xxt} \phi_{xx} \, dx \, dt = 0,$$

for all $\phi \in L^2(0, T; H_0^2(\Omega_t))$; $u(x,0) = u^0(x)$. This result is summarized in the following theorem.

Theorem 4.1. *For each $u^0 \in H_0^2(\Omega_0) \cap H^4(\Omega_0)$, and $\widehat{Q}$ a mixed domain defined as above, there exists a function $u : \widehat{Q} \to \mathbb{R}$ satisfying $u \in L^\infty(0,T; H_0^2(\Omega_t))$, $u_t \in L^\infty(0,T; H_0^2(\Omega_t))$ and*

$$\int_{\widehat{Q}} u_t \phi \, dx \, dt + \int_{\widehat{Q}} (u+u^2)_x \phi \, dx \, dt + \int_{\widehat{Q}} u_{xxt} \phi_{xx} \, dx \, dt = 0,$$

for all $\phi \in L^2(0,T; H_0^2(\Omega_t))$; $u(x,0) = u^0(x)$ for all $x \in \Omega_0$.

In analogous way we have the following result

Theorem 4.2. *For each $u^0 \in H_0^1(\Omega_0) \cap H^2(\Omega_0)$, and $\widehat{Q}$ a mixed domain defined as above, there exists a unique function $u : \widehat{Q} \to \mathbb{R}$, satisfying $u \in L^\infty(0,T; H_0^1(\Omega_t))$, $u_t \in L^\infty(0,T; H_0^1(\Omega_t))$ and*

$$\int_{\widehat{Q}} u_t \phi \, dx \, dt + \int_{\widehat{Q}} (u+u^2)_x \phi \, dx \, dt + \int_{\widehat{Q}} u_{xt} \phi_x \, dx \, dt = 0,$$

for all $\phi \in L^2(0,T; H_0^1(\Omega_t))$; $u(x,0) = u^0(x)$, for all $x \in \Omega_0$.

Acknowledgement. We wish to acknowledge the anonymous referee at the Electronic Journal of Differential Equations, for his constructive remarks and corrections on the original manuscript.

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