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N-TH ORDER IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. We investigate the maximal and minimal solutions of initial value problem for N-th order nonlinear impulsive integro-differential equation in Banach space by establishing a comparison result and using the upper and lower solutions methods.

1. Introduction

The theory of impulsive differential equations in Banach spaces has become an important area of investigation in recent years. In [2], the existence of solution of initial value problem for second order nonlinear impulsive integro-differential equation in Banach space was studied by establishing a comparison result and using the upper and lower solutions methods. Now, in this paper, we shall investigate the existence of solution of initial-value problem (IVP) for N-th order nonlinear impulsive integro-differential equation in Banach space by establishing a new comparison result and using the upper and lower solutions methods. Consider the IVP for impulsive integro-differential equation in a Banach space E:

$$\begin{gathered}
u^{(n)} = f(t, u(t), u'(t), \dots, u^{(n-1)}(t), (Tu)(t)), \quad \forall t \in J, t \neq t_i \\
\Delta u|_{t=t_i} = L_i^0 u^{(n-1)}(t_i) \\
\Delta u'|_{t=t_i} = L_i^1 u^{(n-1)}(t_i) \\
\dots \\
\Delta u^{(n-2)}|_{t=t_i} = L_i^{n-2} u^{(n-1)}(t_i) \\
\Delta u^{(n-1)}|_{t=t_i} = -L_i^{n-1} u^{(n-1)}(t_i) \\
u(0) = u_0,\ u'(0) = u_1, \dots, u^{(n-1)}(0) = u_{n-1}
\end{gathered} \tag{1.1}$$

where $i = 1, 2, \dots, m$, $J = [0, a] (a > 0)$, $u_j \in E (j = 0, 1, 2, \dots, n-1)$, $f \in C[J \times E \times E \times \dots \times E, E]$, $0 < t_1 < \dots < t_i < \dots < t_m < a$, $L_i^j (i = 1, 2, \dots, m;$ $j = 0, 1, \dots, n-1)$ are constants, and

$$(Tu)(t) = \int_0^t k(t, s) u(s) ds, \quad \forall t \in J\,, \tag{1.2}$$

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$k \in C[D, R_+]$, $D = \{(t,s) \in J \times J : t \geq s\}$, R_+ is the set of all nonnegative real numbers, and $k_0 = \max\{k(t,s) : (t,s) \in D\}$. $\Delta u^{(j)}|_{t=t_i}$ denotes the jump of $u^{(j)}(t)$ at $t = t_i$, i.e.

$$\Delta u^{(j)}\big|_{t=t_i} = u^{(j)}(t_i^+) - u^{(j)}(t_i^-),$$

where $u^{(j)}(t_i^+)$ and $u^{(j)}(t_i^-)$ represent the right-hand limit and left-hand limit of $u^{(j)}(t)$ at $t = t_i$ respectively. In (1.1) and the following, $u^{(n-1)}(t_i)$ is understood as $u^{(n-1)}(t_i^-)$.

Let $PC[J,E] = \{u : u$ is a map from J into E such that $u(t)$ is continuous at $t \neq t_i$, left continuous at $t = t_i$ and $u(t_i^+)$ exist for $i = 1,2,\dots,m\}$, $PC^j[J,E] = \{u \in PC^{j-1}[J,E]$: $u^{(j)}(t)$ is continuous at $t \neq t_i$, left continuous at $t = t_i$ and $u^{(j)}(t_i^+)$ exist for $i = 1,2,\dots,m\}$ $(j = 1,2,\dots,n-2)$ and $PC^{n-1}[J,E] = \{u \in PC^{n-2}[J,E] : u^{(n-1)}(t)$ is continuous at $t \neq t_i$, and $u^{(n-1)}(t_i^+)$,$u^{(n-1)}(t_i^-)$ exist for $i = 1,2,\dots,m\}$. Evidently, $PC[J,E]$ is a Banach space with norm

$$\|u\|_{pc} = \sup_{t\in J} \|u(t)\|.$$

It is clear that $PC^j[J,E]$ is a Banach space with norm

$$\|u\|_j = \max\{\|u\|_{pc}, \|u'\|_{pc}, \dots, \|u^{(j)}\|_{pc}\}, \quad (j = 1,2,\dots,n-1)$$

Let $J' = J\backslash\{t_1,\dots,t_m\}$, $\tau = \max\{t_i - t_{i-1} : i = 1,2,\dots,m+1\}$, (where $t_0 = 0, t_{m+1} = a$), $J_0 = [0,t_1]$, $J_1 = (t_1,t_2],\dots,J_{m-1} = (t_{m-1},t_m]$, $J_m = (t_m,a]$. A map $u \in PC^{n-1}[J,E] \bigcap C^n[J',E]$ is called a solution of (1.1) if it satisfies (1.1).

2. Comparison Result

Let E be partially ordered by a cone P of E, i.e. $x \leq y$ if and only if $y - x \in P$. P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N \leq \|y\|$, where θ denotes the zero element of E, and P is said to be regular if $x_1 \leq x_2 \leq \dots \leq x_n \leq \dots \leq y$ implies $\|x_n - x\| \to 0$ as $n \to \infty$ for some $x \in E$. It is well known that the regularity of P implies the normality of P. For details on cone theory, see [1].

Lemma 2.1 (Comparison result). *Assume that $p \in PC^{n-1}[J,E] \bigcap C^n[J',E]$ satisfies*

$$\begin{gathered} p^{(n)}(t) \leq -M_0 p - M_1 p' - M_2 p'' - \dots - M_{n-1}p^{(n-1)} - NTp, \quad \forall t \in J, t \neq t_i \\ \Delta p|_{t=t_i} = L_i^0 p^{(n-1)}(t_i) \\ \Delta p'|_{t=t_i} = L_i^1 p^{(n-1)}(t_i) \\ \dots \\ \Delta p^{(n-2)}|_{t=t_i} = L_i^{n-2} p^{(n-1)}(t_i) \\ \Delta p^{(n-1)}|_{t=t_i} \leq -L_i^{n-1} p^{(n-1)}(t_i), \quad (i = 1,2,\dots,m) \\ p^{(n-1)}(0) \leq p^{(j)}(0) \leq \theta, \quad j = 0,1,\cdots,n-2, \end{gathered} \tag{2.1}$$

where $M_j \geq 0$, $L_i^j \geq 0$ $(j = 0,1,\dots,n-1; i = 1,2,\dots,m)$ are constants and

$$\sum_{i=1}^m L_i^{n-1} + (m+1)M_0\tau \leq 1 \tag{2.2}$$

where

$$M_0 = M_{n-1} + \sum_{i=0}^{n-2} c_i^\star + k_1^\star a + \sum_{j=0}^{n-2}(M_j \sum_{i=1}^{m} L_i^j) + \sum_{j=0}^{n-2}(d_j^\star \sum_{i=1}^{m} L_i^j)a \tag{2.3}$$

$$c_i^* = \sum_{j\le i} \frac{a^{i-j}}{(i-j)!} M_j + \frac{a^{i+1}}{(i+1)!} N k_0, i = 0, 1, \dots, n-2\,,$$

$$k_1^* = \frac{a^{n-2}}{(n-2)!} M_0 + \frac{a^{n-3}}{(n-3)!} M_1 + \dots + M_{n-2} + \frac{k_0 N a^{n-1}}{(n-1)!},$$

$$d_0^* = N k_0,$$

$$d_1^\star = N k_0 a + M_0,$$

$$d_j^* = \frac{N k_0 a^j}{j!} + \frac{M_0 a^{j-1}}{(j-1)!} + \dots + M_{j-2} a + M_{j-1}, j = 2, 3, \dots, n-2,$$

then $p^{(j)}(t) \le \theta, \forall t \in J, (j = 0, 1, \dots, n-1)$, *where* $p^{(0)}(t) = p(t)$.

Proof. Let $p_1(t) = p^{(n-1)}(t), t \in J$. Then $p_1 \in PC[J, E] \bigcap C^1[J', E]$ and

$$p^{(n-2)}(t) = p^{(n-2)}(0) + \int_0^t p_1(s) ds + \sum_{0<t_i<t} [p^{(n-2)}(t_i^+) - p^{(n-2)}(t_i^-)],$$

$$\begin{aligned} p^{(n-3)}(t) = p^{(n-3)}(0) + t p^{(n-2)}(0) + \int_0^t ds_1 \int_0^{s_1} p_1(s_2) ds_2 \\ + \int_0^t \sum_{0<t_i<s} [p^{(n-2)}(t_i^+) - p^{(n-2)}(t_i^-)] ds \\ + \sum_{0<t_i<t} [p^{(n-3)}(t_i^+) - p^{(n-3)}(t_i^-)] \end{aligned}$$

...

$$\begin{aligned} p'(t) &= p'(0) + t p''(0) + \dots + \frac{t^{n-3}}{(n-3)!} p^{(n-2)}(0) \\ &+ \int_0^t ds_1 \int_0^{s_1} ds_2 \dots \int_0^{s_{n-3}} p_1(s_{n-2}) ds_{n-2} \\ &+ \int_0^t ds_1 \int_0^{s_1} ds_2 \dots \int_0^{s_{n-4}} \sum_{0<t_i<s_{n-3}} [p^{(n-2)}(t_i^+) - p^{(n-2)}(t_i^-)] ds_{n-3} \\ &+ \dots + \int_0^t \sum_{0<t_i<s} [p''(t_i^+) - p''(t_i^-)] ds + \sum_{0<t_i<t} [p'(t_i^+) - p'(t_i^-)] \end{aligned}$$

$$\begin{aligned} p(t) &= p(0) + t p'(0) + \dots + \frac{t^{n-2}}{(n-2)!} p^{(n-2)}(0) \\ &+ \int_0^t ds_1 \int_0^{s_1} ds_2 \dots \int_0^{s_{n-2}} p_1(s_{n-1}) ds_{n-1} \\ &+ \int_0^t ds_1 \int_0^{s_1} ds_2 \dots \int_0^{s_{n-3}} \sum_{0<t_i<s_{n-2}} [p^{(n-2)}(t_i^+) - p^{(n-2)}(t_i^-)] ds_{n-2} \end{aligned}$$

$$+\cdots+\int_0^t\sum_{0<t_i<s}[p'(t_i^+)-p'(t_i^-)]ds+\sum_{0<t_i<t}[p(t_i^+)-p(t_i^-)]\,.$$

It is easy to see by induction that for $m=1,2,\dots,n-1$,

$$\int_0^t ds_1\int_0^{s_1}ds_2\dots\int_0^{s_{m-1}}A(s_m)ds_m=\frac{1}{(m-1)!}\int_0^t(t-s)^{m-1}A(s)ds,$$

where A denotes an integrable function. So, we have

$$\begin{aligned}
p^{(n-1)}(t)&=p_1(t)\\
p^{(n-2)}(t)&=p^{(n-2)}(0)+\int_0^t p_1(s)ds+\sum_{0<t_i<t}[p^{(n-2)}(t_i^+)-p^{(n-2)}(t_i^-)]\\
p^{(n-3)}(t)&=p^{(n-3)}(0)+tp^{(n-2)}(0)+\int_0^t(t-s)p_1(s)ds\\
&\quad+\int_0^t\sum_{0<t_i<s}[p^{(n-2)}(t_i^+)-p^{(n-2)}(t_i^-)]ds\\
&\quad+\sum_{0<t_i<t}[p^{(n-3)}(t_i^+)-p^{(n-3)}(t_i^-)]\\
&\dots\\
p'(t)&=p'(0)+tp''(0)+\cdots+\frac{t^{n-3}}{(n-3)!}p^{(n-2)}(0)+\frac{1}{(n-3)!}\int_0^t(t-s)^{n-3}p_1(s)ds\\
&\quad+\frac{1}{(n-4)!}\int_0^t(t-s)^{n-4}\sum_{0<t_i<s}[p^{(n-2)}(t_i^+)-p^{(n-2)}(t_i^-)]ds\\
&\quad+\cdots+\int_0^t\sum_{0<t_i<s}[p''(t_i^+)-p''(t_i^-)]ds+\sum_{0<t_i<t}[p'(t_i^+)-p'(t_i^-)]\\
p(t)&=p(0)+tp'(0)+\cdots+\frac{t^{n-2}}{(n-2)!}p^{(n-2)}(0)+\frac{1}{(n-2)!}\int_0^t(t-s)^{n-2}p_1(s)ds\\
&\quad+\frac{1}{(n-3)!}\int_0^t(t-s)^{n-3}\sum_{0<t_i<s}[p^{(n-2)}(t_i^+)-p^{(n-2)}(t_i^-)]ds\\
&\quad+\cdots+\int_0^t\sum_{0<t_i<s}[p'(t_i^+)-p'(t_i^-)]ds+\sum_{0<t_i<t}[p(t_i^+)-p(t_i^-)]
\end{aligned}\tag{2.4}$$

Substituting (2.4) into (2.1), we get

$$\begin{aligned}
p_1'(t)&\le -M_{n-1}p_1(t)-c_0(t)p(0)-c_1(t)p'(0)-\cdots-c_{n-2}(t)p^{(n-2)}(0)\\
&\quad-\int_0^t k_1(t,s)p_1(s)ds-\sum_{j=0}^{n-2}M_j\sum_{0<t_i<t}[p^{(j)}(t_i^+)-p^{(j)}(t_i)]\\
&\quad-\sum_{j=0}^{n-2}\int_0^t d_j(t,s)\sum_{0<t_i<s}[p^{(j)}(t_i^+)-p^{(j)}(t_i)]ds,\quad\forall t\in J
\end{aligned}\tag{2.5}$$

Where

$$\begin{gathered}
c_0(t)=M_0+N\int_0^t k(t,s)ds,\\
c_1(t)=tM_0+M_1+N\int_0^t sk(t,s)ds,\\
\dots\\
c_{n-2}(t)=\frac{t^{n-2}}{(n-2)!}M_0+\frac{t^{n-3}}{(n-3)!}M_1+\dots+M_{n-2}+\frac{N}{(n-2)!}\int_0^t s^{n-2}k(t,s)ds,\\
k_1(t,s)=\frac{(t-s)^{n-2}}{(n-2)!}M_0+\frac{(t-s)^{n-3}}{(n-3)!}M_1\\
+\dots+M_{n-2}+\frac{N}{(n-2)!}\int_s^t (r-s)^{n-2}k(t,r)dr,\\
d_0(t,s)=Nk(t,s),\\
d_1(t,s)=N\int_s^t k(t,r)dr+M_0,\\
d_2(t,s)=\frac{N}{1!}\int_s^t k(t,r)(r-s)dr+M_0(t-s)+M_1,\\
\dots\\
d_{n-2}(t,s)=\frac{N}{(n-3)!}\int_s^t k(t,r)(r-s)^{n-3}dr+M_0\frac{(t-s)^{n-3}}{(n-3)!}\\
+\dots+M_{n-4}(t-s)+M_{n-3}\,.
\end{gathered}$$

For $g\in P^*$, the dual cone of P, let $v(t)=g(p_1(t))$, then $v\in PC[J,R]\bigcap C'[J',R]$. By (2.5) and (2.1), we have

$$\begin{gathered}
v'(t)\le -M_{n-1}v(t)-\sum_{j=0}^{n-2}c_j(t)g(p^{(j)}(0))-\int_0^t k_1(t,s)v(s)ds\\
-\sum_{j=0}^{n-2}M_j\sum_{0<t_i<t}g(p^{(j)}(t_i^+)-p^{(j)}(t_i))\\
-\sum_{j=0}^{n-2}\int_0^t d_j(t,s)\sum_{0<t_i<s}g(p^{(j)}(t_i^+)-p^{(j)}(t_i))ds,\quad \forall t\in J\\
g(p^{(j)}(t_i^+)-p^{(j)}(t_i))=L_i^j v(t_i)\quad (j=0,1,2,\dots,n-2;i=1,2,\dots,m)\\
\Delta v|_{t=t_i}\le -L_i^{n-1}v(t_i)\quad (i=1,2,\dots,m)\\
v(0)\le g(p^{(j)}(0))\le 0,\quad j=0,1,\cdots,n-2,
\end{gathered}\tag{2.6}$$

We now show that

$$v(t)\le 0,\quad \forall t\in J. \tag{2.7}$$

Assume that (2.7) is not true, i.e. there exists a $0<t^*\le a$ such that $v(t^*)>0$. Let $t^*\in J_j=(t_j,t_{j+1}]$ and $\inf_{0\le t\le t^*}v(t)=-\lambda$. We have $\lambda\ge 0$. Assume that there exist a $J_k=(t_k,t_{k+1}](k\le j)$ such that $v(t_*)=-\lambda$ hold for some $t_*\in J_k$ or $v(t_k^+)=-\lambda$. We may assume that $v(t_*)=-\lambda$, since for the case $v(t_k^+)=-\lambda$ the proof is similar. By (2.6), we have

$$\begin{aligned} v'(t) &\le \lambda[M_{n-1} + \sum_{j=0}^{n-2} c_j(t) + \int_0^t k_1(t,s)ds + \sum_{j=0}^{n-2}(M_j \sum_{i=1}^{m} L_i^j) \\ &\quad + \sum_{j=0}^{n-2} \int_0^t (d_j(t,s) \sum_{0<t_i<s} L_i^j ds] \\ &\le \lambda M_0, \quad \forall t \in [0,t^*], \end{aligned} \tag{2.8}$$

where M_0 is given by (2.3),

$$\begin{gathered} \Delta v|_{t=t_i} \le -L_i^{n-1} v(t_i) \le \lambda L_i^{n-1} \quad (i = 1,2,\dots,m) \\ v(0) \le 0 \end{gathered} \tag{2.9}$$

Now, the mean value theorem implies

$$\begin{gathered} v(t^*) - v(t_j^+) = v'(\xi_j)(t^* - t_j), \quad t_j < \xi_j < t^*; \\ v(t_j) - v(t_{j-1}^+) = v'(\xi_{j-1})(t_j - t_{j-1}), \quad t_{j-1} < \xi_{j-1} < t_j; \\ \dots \\ v(t_{k+2}) - v(t_{k+1}^+) = v'(\xi_{k+1})(t_{k+2} - t_{k+1}), \quad t_{k+1} < \xi_{k+1} < t^{k+2}; \\ v(t_{k+1}) - v(t_*) = v'(\xi_k)(t_{k+1} - t_*), \quad t_* < \xi_k < t_{k+1}. \end{gathered} \tag{2.10}$$

By (2.9) we have

$$v(t_i^+) = v(t_i) + \Delta v|_{t=t_i} \le v(t_i) + \lambda L_i^{n-1}, \quad \forall t_i \le t^*. \tag{2.11}$$

By (2.8), (2.10), (2.11), we obtain

$$\begin{gathered} v(t^*) - v(t_j) - \lambda L_j^{n-1} \le \lambda M_0 \tau \\ v(t_j) - v(t_{j-1}) - \lambda L_{j-1}^{n-1} \le \lambda M_0 \tau \\ \dots \\ v(t_{k+2}) - v(t_{k+1}) - \lambda L_{k+1}^{n-1} \le \lambda M_0 \tau \\ v(t_{k+1}) + \lambda \le \lambda M_0 \tau\,. \end{gathered} \tag{2.12}$$

Adding these inequalities, we obtain

$$v(t^*) + \lambda - \lambda \sum_{i=k+1}^{j} L_i^{n-1} \le (j-k+1)\lambda M_0 \tau$$

and so

$$0 < v(t^*) \le -\lambda + \lambda \sum_{i=1}^{m} L_i^{n-1} + (m+1)\lambda M_0 \tau$$

Evidently $\lambda \neq 0$, so, $\lambda > 0$, then, we have

$$1 < \sum_{i=1}^{m} L_i^{n-1} + (m+1) M_0 \tau,$$

which contradicts (2.2), hence $v(t) \le 0, \forall t \in J$. Since $g \in P^*$ is arbitrary, it implies that $p^{(n-1)}(t) \le \theta$ for all $t \in J$. By $p^{(n-2)}(0) \le \theta$, $\Delta p^{(n-2)}|_{t=t_i} = L_i^{n-2} p^{(n-1)}(t_i) \le \theta$; this implies $p^{(n-2)}(t) \le \theta$ for all $t \in J$. Continuing in this manner, $p^{(i)}(t) \le \theta$ for all $t \in J$, $i = 0,1,\dots,n-3$. The proof is complete. □

Lemma 2.2. *Assume $\sigma \in PC[J,E]$, and M_j, N, L_i^j $(j = 0,1,2,\dots,n-1; i = 1,2,\dots,m)$ are constants, then $u \in PC^{n-1}[J,E]\bigcap C^n[J',E]$ is a solution of the linear IVP*

$$\begin{gathered} u^{(n)} = -\sum_{j=0}^{n-1} M_j u^{(j)} - NTu + \sigma(t), \quad \forall t \in J,\ t \neq t_i \\ \Delta u^{(j)}|_{t=t_i} = L_i^j u^{(n-1)}(t_i), \quad (j = 0,1,\dots,n-2) \\ \Delta u^{(n-1)}|_{t=t_i} = -L_i^{n-1} u^{(n-1)}(t_i), \quad (i = 1,2,\dots,m) \\ u(0) = u_0,\ u'(0) = u_1,\dots,u^{(n-1)}(0) = u_{n-1}. \end{gathered} \tag{2.13}$$

if and only if $u \in PC^{n-1}[J,E]$ is a solution of the linear impulsive integral equation

$$\begin{aligned} u(t) &= u_0 + tu_1 + \frac{t^2}{2!}u_2 + \dots + \frac{t^{n-1}}{(n-1)!}u_{n-1} \\ &+ \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}[-\sum_{j=0}^{n-1} M_j u^{(j)}(s) - N(Tu)(s) + \sigma(s)]ds \\ &+ \sum_{0<t_i<t}[-\frac{(t-t_i)^{n-1}}{(n-1)!}L_i^{n-1} + \frac{(t-t_i)^{n-2}}{(n-2)!}L_i^{n-2} \\ &+ \dots + (t-t_i)L_i^1 + L_i^0]u^{(n-1)}(t_i) \quad \forall t \in J\,. \end{aligned} \tag{2.14}$$

The proof of this lemma is similar to the proof of Lemma 3 in [2]; therefore, we omit it.

Lemma 2.3. *Let $\sigma \in PC[J,E]$, $M_j \geq 0$, $N \geq 0$, $L_i^j \geq 0$ $(j = 0,1,2,\dots,n-1; i = 1,2,\dots,m)$ be constants. Assume*

$$\begin{gathered} \beta_j = \frac{\sum_{i=0}^{n-1} M_i + Nk_0 a}{(n-j)!} a^{n-j} + \sum_{i=1}^{m}[\frac{(a-t_i)^{n-j-1}}{(n-j-1)!}L_i^{n-1} \\ + \dots + (a-t_i)L_i^{j+1} + L_i^j] < 1 \\ \beta = \max_j\{\beta_j\} \end{gathered} \tag{2.15}$$

where $j = 0,1,\dots,n-1$). Then the impulsive integral equation (2.14) has a unique solution in $PC^{n-1}[J,E]$.

Proof. Define operator F by

$$\begin{aligned} (Fu)(t) =& u_0 + tu_1 + \frac{t^2}{2!}u_2 + \dots + \frac{t^{n-1}}{(n-1)!}u_{n-1} \\ &+ \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}[-\sum_{j=0}^{n-1} M_j u^{(j)}(s) - N(Tu)(s) + \sigma(s)]ds \\ &+ \sum_{0<t_i<t}[-\frac{(t-t_i)^{n-1}}{(n-1)!}L_i^{n-1} + \frac{(t-t_i)^{n-2}}{(n-2)!}L_i^{n-2} \\ &+ \dots + (t-t_i)L_i^1 + L_i^0]u^{(n-1)}(t_i) \quad (\forall t \in J). \end{aligned}$$

Then for all $t \in J', j = 1, 2, \dots, n-1$,

$$
\begin{aligned}
(Fu)^{(j)}(t) &= u_j + tu_{j+1} + \dots + \frac{t^{n-j-1}}{(n-j-1)!}u_{n-1} \\
&\quad + \frac{1}{(n-j-1)!}\int_0^t (t-s)^{n-j-1}[-\sum_{j=0}^{n-1} M_j u^{(j)}(s) - N(Tu)(s) + \sigma(s)]ds \\
&\quad + \sum_{0<t_i<t} [-\frac{(t-t_i)^{n-j-1}}{(n-j-1)!} L_i^{n-1} + \dots + (t-t_i)L_i^{j+1} + L_i^j] u^{(n-1)}(t_i)
\end{aligned}
$$

and $F : PC^{n-1}[J, E] \to PC^{n-1}[J, E]$. For $u, v \in PC^{n-1}[J, E]$, by (2.12) we have

$$
\begin{aligned}
&\|(Fu)^{(j)}(t) - (Fv)^{(j)}(t)\| \\
&\le \frac{\sum_{i=0}^{n-1} M_i + Nk_0 a}{(n-j-1)!}\|u-v\|_{n-1} \int_0^t (t-s)^{n-j-1} + \sum_{i=1}^{m}[\frac{(t-t_i)^{n-j-1}}{(n-j-1)!} L_i^{n-1} \\
&\quad + \frac{(t-t_i)^{n-j-2}}{(n-j-2)!} L_i^{n-2} + \dots + (t-t_i)L_i^{j+1} + L_i^j]\|u-v\|_{n-1} \\
&\le \beta_j \|u-v\|_{n-1} \quad (\forall t \in J, j = 0, 1, \dots, n-1)
\end{aligned}
$$

and

$$
\|Fu - Fv\|_{n-1} \le \beta\|u-v\|_{n-1}, \quad \forall u, v \in PC^{n-1}[J, E] \tag{2.16}
$$

where β_j, β is defined by (2.15), The Banach fixed point implies that F has a unique fixed point in $PC^{n-1}[J, E]$, and the lemma is proved. □

3. Main Theorem

Let us list some conditions used for stating the main result.

(H1) There exist $v_0, w_0 \in PC^{n-1}[J, E] \bigcap C^n[J', E]$ with $v_0(t) \le w_0(t) (t \in J)$ such that

$$
\begin{gathered}
v_0^{(n)} \le f(t, v_0, v_0', \dots, v_0^{(n-1)}, Tv_0), \quad \forall t \in J, t \ne t_i \\
\Delta v_0^{(j)}|_{t=t_i} = L_i^j v_0^{(n-1)}(t_i), \quad (j = 0, 1, \dots, n-2; i = 1, 2, \dots, m) \\
\Delta v_0^{(n-1)}|_{t=t_i} \le -L_i^{n-1} v_0^{(n-1)}(t_i) \\
v_0^{(j)}(0) \le u_j, v_0^{(n-1)}(0) - v_0^{(j)}(0) \le u_{n-1} - u_j \quad (j = 0, 1, 2, \dots, n-1),
\end{gathered}
$$

and

$$
\begin{gathered}
w_0^{(n)} \ge f(t, w_0, w_0', \dots, w_0^{(n-1)}, Tw_0), \quad \forall t \in J, t \ne t_i \\
\Delta w_0^{(j)}|_{t=t_i} = L_i^j w_0^{(n-1)}(t_i), \quad (j = 0, 1, \dots, n-2; i = 1, 2, \dots, m) \\
\Delta w_0^{(n-1)}|_{t=t_i} \ge -L_i^{n-1} w_0^{(n-1)}(t_i) \\
w_0^{(j)}(0) \ge u_j, \quad w_0^{(n-1)}(0) - w_0^{(j)}(0) \ge u_{n-1} - u_j, \quad (j = 0, 1, 2, \dots, n-1),
\end{gathered}
$$

where $L_i^j \ge 0$, $(i = 1, 2, \dots, m; j = 0, 1, \dots, n-1)$, v_0 and w_0 are lower and upper solution of (1.1) respectively.

(H2) There exist constants $M_i \ge 0$ $(i = 0, 1, \dots, n-1)$ and $N \ge 0$ such that

$$
f(t, u_0, u_1, u_2, \dots, u_{n-1}, v) - f(t, \bar{u}_0, \bar{u}_1, \bar{u}_2, \dots, \bar{u}_{n-1}, \bar{v})
$$

$$
\begin{gathered}
\geq -\sum_{j=0}^{n-1} M_j(u_j - \bar{u}_j) - N(v - \bar{v}), \quad \forall t \in J \\
v_0^{(j)} \leq \bar{u}_j \leq u_j \leq w_0^{(j)}, \quad (j = 0, 1, \dots, n-1) \\
Tv_0 \leq \bar{v} \leq v \leq Tw_0
\end{gathered}
$$

Let $[v_0, w_0] = \{u \in PC^{n-1}[J, E] : v_0^{(j)}(t) \leq u^{(j)}(t) \leq w_0^{(j)}(t), t \in J, j = 0, 1, \dots, n-1\}$

Theorem 3.1. *Let cone P be regular and f be uniformly continuous on $J \times B_r \times B_r \times \cdots \times B_r$ for any $r > 0$, where $B_r = \{x \in E : \|x\| \leq r\}$. Suppose that conditions (H1) and (H2) are satisfied, $L_i^j \geq 0 (j = 0, 1, \dots, n-1; i = 1, 2, \dots, m)$ and inequalities (2.2), (2.15) hold. Then* (1.1) *has minimal and maximal solutions $\bar{u}$ and u^* in $[v_0, w_0]$; Moreover, there exist monotone sequences $\{v_k(t)\}$ and $\{w_k(t)\}$ such that $\{v_k^{(j)}(t)\}$,$\{w_k^{(j)}(t)\}(j = 0, 1, 2, \dots, n-1)$ converge uniformly on $J_j(j = 0, 1, \dots, m)$ to the $\bar{u}^{(j)}(t)$ and $(u^*)^{(j)}(t)(j = 0, 1, 2, \dots, n-1)$ respectively, and*

$$
\begin{gathered}
v_0^{(j)}(t) \leq v_1^{(j)}(t) \leq \cdots \leq v_k^{(j)}(t) \leq \cdots \leq \bar{u}^{(j)}(t) \\
\leq u^{(j)}(t) \leq (u^*)^{(j)}(t) \leq \cdots \leq w_k^{(j)}(t) \leq \cdots \leq w_1^{(j)}(t) \leq w_0^{(j)}(t)
\end{gathered}
\tag{3.1}
$$

for all $t \in J$, $j = 0, 1, \dots, n-1$, where $u(t)$ is any solution of (1.1) *in* $[v_0, w_0]$.

Proof. For $\eta \in [v_0, \omega_0]$, consider the linear problem (2.13) with

$$
\sigma(t) = f(t, \eta(t), \eta'(t), \dots, \eta^{(n-1)}(t), (T\eta)(t)) + \sum_{j=0}^{n-1} M_j \eta^{(j)}(t) + N(T\eta)(t) \tag{3.2}
$$

By Lemma 2.3, (2.13) has a unique solution $u \in PC^{n-1}[J, E]$. Let $u = A\eta$. Then $A : [v_0, w_0] \to PC^{n-1}[J, E] \bigcap C^n[J', E] \subset PC[J, E]$, we now show that

- (a) $v_0^{(j)}(t) \leq (Av_0)^{(j)}(t), (Aw_0)^{(j)}(t) \leq (w_0)^{(i)}(t)$, $t \in J, j = 0, 1, 2, \dots, n-1$
- (b) $\eta_1, \eta_2 \in [v_0, w_0]$, $\eta_1^{(j)} \leq \eta_2^{(j)}$ implies $(A\eta_1)^{(j)} \leq (A\eta_2)^{(j)}$, $t \in J$, $j = 0, 1, 2, \dots, n-1$.

To prove (a), we set $v_1 = Av_0$ and $p = v_0 - v_1$. From (2.13) and (3.2), we have

$$
\begin{gathered}
v_1^{(n)} = f(t, v_0, v_0', \dots, v_0^{(n-1)}, Tv_0) + \sum_{j=0}^{n-1} M_j v_0^{(j)} + N(Tv_0) \\
-\sum_{j=0}^{n-1} M_j v_1^{(j)} - N(Tv_1), \quad \forall t \in J, t \neq t_i \\
\Delta v_1^{(j)}|_{t=t_i} = L_i^j v_1^{(n-1)}(t_i), \quad (j = 0, 1, \dots, n-2) \\
\Delta v_1^{(n-1)}|_{t=t_i} = -L_i^{n-1} v_1^{(n-1)}(t_i), \quad (i = 1, 2, \dots, m) \\
v_1(0) = u_0,\ v_1'(0) = u_1, \dots, v_1^{(n-1)}(0) = u_{n-1}
\end{gathered}
$$

so, by (H1),

$$
\begin{gathered}
p^{(n)}(t) \leq -\sum_{j=0}^{n-1} M_j p^{(j)}(t) - N(Tp)(t), \quad \forall t \in J, t \neq t_i, \\
\Delta p^{(j)}|_{t=t_i} = L_i^j p^{(n-1)}(t_i), \quad (j = 0, 1, \dots, n-2; i = 1, 2, \dots, m),
\end{gathered}
$$

$$
\begin{gathered}
\Delta p^{(n-1)}|_{t=t_i} \le -L_i^{n-1} p^{(n-1)}(t_i), \quad (i=1,2,\dots,m),\\
p^{(n-1)}(0) \le p^{(j)}(0) \le \theta, \quad (j=0,1,2,\dots,n-2),
\end{gathered}
$$

which implies by virtue of Lemma 2.1 that $p^{(j)}(t) \le \theta$ $(j=0,1,\dots,n-1)$ for $t \in J$, i.e. $v_0^{(j)}(t) \le (Av_0)^{(j)}(t)$, for all $t \in J$, $j=0,1,2,\dots,n-1$. Similarly, we can show that $(Aw_0)^{(j)}(t) \le (w_0)^{(j)}(t)$ for all $t \in J$, $j=0,1,2,\dots,n-1$.

To prove (b), let $\eta_1, \eta_2 \in [v_0, w_0]$, such that $\eta_1^{(j)} \le \eta_2^{(j)}$ and $p = A\eta_1 - A\eta_2$. Then, from (2.13) and (H2), we have

$$
\begin{gathered}
p^{(n)}(t) \le -\sum_{j=0}^{n-1} M_j p^{(j)}(t) - N(Tp)(t), \quad \forall t \in J, t \ne t_i,\\
\Delta p^{(j)}|_{t=t_i} = L_i^j p^{(n-1)}(t_i), \quad (j=0,1,\dots,n-2; i=1,2,\dots,m),\\
\Delta p^{(n-1)}|_{t=t_i} = -L_i^{n-1} p^{(n-1)}(t_i), \quad (i=1,2,\dots,m),\\
p^{(j)}(0) = \theta, \quad (j=0,1,2,\dots,n-1).
\end{gathered}
$$

So, Lemma 2.1 implies (b). Let

$$
v_k = Av_{k-1}, \quad w_k = Aw_{k-1}, \quad k=1,2,\dots, \tag{3.3}
$$

By (a) and (b) above, we have

$$
v_0^{(j)}(t) \le v_1^{(j)}(t) \le \dots \le v_k^{(j)}(t) \le \dots \le w_k^{(j)}(t) \le \dots \le w_1^{(j)}(t) \le w_0^{(j)}(t), \tag{3.4}
$$

for all $t \in J, j=0,1,2,\dots,n-1$. On account of the definition of v_k, we have

$$
\begin{aligned}
v_k(t) &= u_0 + tu_1 + \frac{t^2}{2!}u_2 + \dots + \frac{t^{n-1}}{(n-1)!}u_{n-1}\\
&\quad + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}[-\sum_{j=0}^{n-1} M_j v_k^{(j)}(s) - N(Tv_k)(s) + \sigma_{k-1}(s)]ds\\
&\quad + \sum_{0<t_i<t}[-\frac{(t-t_i)^{n-1}}{(n-1)!}L_i^{n-1} + \frac{(t-t_i)^{n-2}}{(n-2)!}L_i^{n-2}\\
&\quad + \dots + (t-t_i)L_i^1 + L_i^0]v_k^{(n-1)}(t_i), \quad (\forall t \in J, k=1,2,3,\dots)
\end{aligned} \tag{3.5}
$$

where

$$
\begin{aligned}
\sigma_{k-1}(t) &= f(t, v_{k-1}(t), v_{k-1}'(t), \dots, v_{k-1}^{(n-1)}(t), (Tv_{k-1})(t))\\
&\quad + \sum_{j=0}^{n-1} M_j v_{k-1}^{(j)}(t) + N(Tv_{k-1})(t), \quad \forall t \in J, \quad k=1,2,3,\dots
\end{aligned} \tag{3.6}
$$

so,

$$
\begin{aligned}
&v_k^{(j)}(t)\\
&= u_j + tu_{j+1} + \dots + \frac{t^{n-j-1}}{(n-j-1)!}u_{n-1}\\
&\quad + \frac{1}{(n-j-1)!}\int_0^t (t-s)^{n-j-1}[-\sum_{j=0}^{n-1} M_j v_k^{(j)}(s) - N(Tv_k)(s) + \sigma_{k-1}(s)]ds\\
&\quad + \sum_{0<t_i<t}[-\frac{(t-t_i)^{n-j-1}}{(n-j-1)!}L_i^{n-1} + \frac{(t-t_i)^{n-j-2}}{(n-j-2)!}L_i^{n-2}\\
&\quad + \dots + (t-t_i)L_i^{j+1} + L_i^j]v_k^{(n-1)}(t_i),
\end{aligned}
\tag{3.7}
$$

for all $t \in J'$, $j = 1, 2, \dots, n-1$, $k = 1, 2, 3, \dots$. Similar to the (2.16), for $k, i = 1, 2, \dots,$ we can obtain

$$
\|v_{k+i} - v_k\|_{n-1} \le \beta\|v_{k+1} - v_k\|_{n-1} + \beta^*\|\sigma_{k+i-1} - \sigma_{k-1}\|_{pc},
$$

where β is defined by (2.15) and

$$
\beta^* = \max\{\frac{a^n}{n!}, \frac{a^{n-1}}{(n-1)!}, \dots, \frac{a^2}{2}, a\}\,. \tag{3.8}
$$

Hence, for $k, i = 1, 2, \dots,$

$$
\|v_{k+i} - v_k\|_{n-1} \le \frac{\beta^*}{1-\beta}\|\sigma_{k+i-1} - \sigma_{k-1}\|_{pc}\,. \tag{3.9}
$$

Since the regularity of P implies the normality of P, we see from (3.4) that $V_j = \{v_k^{(j)} : k = 0, 1, 2, \dots\}$ $(j = 0, 1, \dots, n-1)$ is a bounded set in $PC^j[J, E]$. It is easy to show that the uniform continuity of f on $J \times B_r \times B_r \times \dots \times B_r$ implies the boundedness of f on $J \times B_r \times B_r \times \dots \times B_r$, so by (3.6),there is a constant $b > 0$ such that

$$
\|\sigma_{k-1}\|_{pc} \le b \quad (k = 1, 2, \dots)
$$

and therefore, from (3.7) we know that functions $\{v_k^{(j)}(t)\}$ $(j = 0, 1, \dots, n-2)$ are equicontinuous on each J_i $(i = 0, 1, \dots, m)$. From (3.4) and the regularity of P, we can infer that $\{v_k^{(j)}(t)\}$ converges uniformly to $\bar{u}^{(j)}(t) \in PC[J, E]$ in J; i.e.,

$$
\|v_k^{(j)} - \bar{u}^{(j)}\|_{pc} \to 0 \quad (k \to \infty) \tag{3.10}
$$

From (3.6),(3.10) and the uniform continuity of f on $J \times B_r \times B_r \times \dots \times B_r$, we get

$$
\|\sigma_{k-1} - \bar{\sigma}\|_{pc} \to 0 \quad (k \to \infty)
$$

where

$$
\bar{\sigma}(t) = f(t, \bar{u}(t), \bar{u}'(t), \dots, \bar{u}^{(n-1)}(t), (T\bar{u})(t)) + \sum_{j=0}^{n-1} M_i \bar{u}^{(j)}(t) + N(T\bar{u})(t),
$$

for all $t \in J$. Taking limits in (3.5),we obtain

$$
\bar{u}(t) = u_0 + tu_1 + \frac{t^2}{2!}u_2 + \dots + \frac{t^{n-1}}{(n-1)!}u_{n-1}
$$

$$
\begin{aligned}
&+\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}[-\sum_{j=0}^{n-1} M_j \bar{u}^{(j)}(s) - N(T\bar{u})(s) + \sigma_{k-1}(s)]ds \\
&+\sum_{0<t_i<t}[-\frac{(t-t_i)^{n-1}}{(n-1)!}L_i^{n-1} + \frac{(t-t_i)^{n-2}}{(n-2)!}L_i^{n-2} \\
&+\cdots+(t-t_i)L_i^1 + L_i^0]\bar{u}^{(n-1)}(t_i)
\end{aligned}
$$

which by Lemma 2.2 implies $\bar{u} \in PC^{n-1}[J,E]\bigcap C^n[J',E]$ and $\bar{u}(t)$ is a solution of (1.1).

In the same way, we can show that $\|w_k - u^*\|_{n-1} \to 0(k \to \infty)$ for some $u^* \in PC^{n-1}[J,E]\bigcap C^n[J',E]$ and $u^*(t)$ is a solution of (1.1).

Finally, let $u \in PC^{n-1}[J,E]\bigcap C^n[J',E]$ be any solution of (1.1) in $[v_0, w_0]$ satisfying $v_0^{(j)}(t) \le u^{(j)}(t) \le w_0^{(j)}(t)$, for all $t \in J$, $j = 0,1,\dots,n-1$. Assume that $v_{m-1}^{(j)}(t) \le u^{(j)}(t) \le w_{m-1}^{(j)}(t)$ for all $t \in J$, $j = 0,1,\dots,n-1$. Then by (H2) and Lemma 2.1, we can infer that $v_m^{(j)}(t) \le u^{(j)}(t) \le w_m^{(j)}(t)$ for all $t \in J$, $j = 0,1,\dots,n-1$. Hence, by induction, $v_k^{(j)}(t) \le u^{(j)}(t) \le w_k^{(j)}(t)$ for all $t \in J$, $j = 0,1,\dots,n-1$, $k = 1,2,\dots$, which implies that $\bar{u}^{(j)}(t) \le u^{(j)}(t) \le (u^*)^{(j)}(t)$ for all $t \in J$, $j = 0,1,\dots,n-1$. Hence (3.1) follows from (3.4). The proof is complete. □

Remark 3.2. The condition that P is regular will be satisfied if E is weakly complete (reflexive, in particular) and P is normal (see [4, theorem 2]).

4. An Example

Consider the initial-value problem infinite system for scalar third order integro-differential equations

$$
\begin{aligned}
u_n^{(3)} &= \frac{1}{100n^2}[(t-u_n)^2 + t^2 u_{n+1} + (u_{2n}')^2 + (t-u_n'')^2] \\
&\quad + \frac{t}{800n^3}(t - \int_0^t e^{-ts}u_n(s)ds)^2, \quad 0 \le t \le 2, t \ne 1 \\
\Delta u_n|_{t=1} &= \frac{1}{3}u_n''(1) \\
\Delta u_n'|_{t=1} &= \frac{1}{2}u_n''(1) \\
\Delta u_n''|_{t=1} &= -\frac{1}{60}u_n''(1) \\
u_n(0) = u_n'(0) &= u_n''(0) = 0, \quad n = 1,2,\dots
\end{aligned}
\tag{4.1}
$$

Claim: The system (4.1) admits minimal and maximal solutions which are continuously differentiable on $[0,1)\cup(1,2]$ and satisfy

$$
0 \le u_n(t) \le \begin{cases} t^3/n^2, & if 0 \le t \le 1 \\ (t^3+t^2+t)/n^2, & if 1 < t \le 2 \end{cases}
$$

$$
0 \le u_n'(t) \le \begin{cases} 3t^2/n^2, & if 0 \le t \le 1 \\ (3t^2+2t+1)/n^2, & if 1 < t \le 2 \end{cases}
$$

$$0 \le u_n''(t) \le \begin{cases} 6t/n^2, & if 0 \le t \le 1 \\ (6t+2)/n^2, & if 1 < t \le 2 \end{cases}$$

where $n = 1, 2, 3, \dots$.

Proof. Let $E = l^1 = \{u = (u_1, u_2, \dots, u_n, \dots) : \sum_{n=1}^{\infty} |u_n| < \infty\}$ with the norm $\|u\| = \sum_{n=1}^{\infty} |u_n|$ and let $P = \{u = (u_1, u_2, \dots, u_n, \dots) \in l^1 : u_n \ge 0, n = 1, 2, \dots\}$. Then P is a normal cone in E. Since l^1 is weakly complete, from remark 3.1 we know that P is regular. System (4.1) can be regarded as an IVP of form (1.1), where

$$\begin{gathered} a = 2, k(t,s) = e^{-ts}, \quad u = (u_1, u_2, \dots, u_n, \dots), \\ v = (v_1, v_2, \dots, v_n, \dots), \quad w = (w_1, w_2, \dots, w_n, \dots), \\ z = (z_1, z_2, \dots, z_n, \dots), \quad f = (f_1, f_2, \dots, f_n, \dots), \end{gathered}$$

in which

$$f_n(t,u,v,w,z) = \frac{1}{100n^2}[(t-u_n)^2 + t^2 u_{n+1} + (v_{2n})^2 + (t-w_n)^2] + \frac{t}{800n^3}(t-z_n)^2 \tag{4.2}$$

and $m = 1$, $t_1 = 1$, $L_1^0 = \frac{1}{3}$, $L_1^1 = \frac{1}{2}$, $L_1^2 = \frac{1}{60}$, $u_0 = u_1 = u_2 = (0, 0, \dots, 0, \dots)$.

Evidently, $f \in C[J \times E \times E \times E \times E, E]$ $(J = [0,2])$. Let $v_0(t) = (0, 0, \dots, 0, \dots)$, for $0 \le t \le 2$ and

$$w_0(t) = \begin{cases} (t^3, \dots, t^3/n^2, \dots), & if 0 \le t \le 1; \\ (t^3 + t^2 + t, \dots, \frac{t^3+t^2+t}{n^2}, \dots), & if 1 < t \le 2. \end{cases}$$

We have $v_0 \in C^3[J,E]$, $w_0 \in PC^2[J,E] \cap C^3[J',E]$, where $J' = J \backslash \{1\} = [0,1) \cup (1,2]$, $v_0(t) \le w_0(t)$ $(t \in J)$ and

$$w_0'(t) = \begin{cases} (3t^2, \dots, \frac{3t^2}{n^2}, \dots), & if 0 \le t \le 1 \\ (3t^2 + 2t + 1, \dots, \frac{3t^2+2t+1}{n^2}, \dots), & if 1 < t \le 2 \end{cases}$$

$$w_0''(t) = \begin{cases} (6t, \dots, \frac{6t}{n^2}, \dots), & if 0 \le t \le 1 \\ (6t+2, \dots, \frac{6t+2}{n^2}, \dots), & if 1 < t \le 2 \end{cases}$$

$$w_0^{(3)} = (6, \dots, \frac{6}{n^2}, \dots), \quad \forall 0 \le t \le 2.$$

It is clear that

$$\begin{gathered} v_0'(t) \le w_0'(t), v_0''(t) \le w_0''(t), \quad \forall t \in J \\ v_0(0) = w_0(0) = (0, 0, \dots, 0, \dots) = u_0, \\ v_0'(0) - v_0(0) = w_0'(0) - w_0(0) = u_1 - u_0 = (0, 0, \dots, 0, \dots) \\ v_0''(0) - v_0'(0) = w_0''(0) - w_0'(0) = u_2 - u_1 = (0, 0, \dots, 0, \dots) \\ v_0^{(3)}(t) = (0, 0, \dots, 0, \dots), \quad \forall t \in J \\ \Delta v_0|_{t=1} = (0, 0, \dots, 0, \dots) = \frac{1}{3} v_0''(1) \\ \Delta v_0'|_{t=1} = (0, 0, \dots, 0, \dots) = \frac{1}{2} v_0''(1) \\ \Delta v_0''|_{t=1} = (0, 0, \dots, 0, \dots) = -\frac{1}{60} v_0''(1) \end{gathered}$$

$$\begin{gathered}\Delta w_0|_{t=1} = (2,\dots,\frac{2}{n^2},\dots) = \frac{1}{3}w_0''(1)\\ \Delta w_0'|_{t=1} = (3,\dots,\frac{3}{n^2},\dots) = \frac{1}{2}w_0''(1)\\ \Delta w_0''|_{t=1} = (2,\dots,\frac{2}{n^2},\dots) > -\frac{1}{60}w_0''(1)\\ f_n(t,v_0(t),v_0'(t),v_0''(t),(Tv_0)(t)) = \frac{2t^2}{100n^2}+\frac{t^3}{800n^3} \geq 0 = v_0^{(3)}(t), \quad \forall t\in J.\end{gathered}$$

When $0\leq t\leq 1$,we have

$$\begin{aligned}&f_n(t,w_0(t),w_0'(t),w_0''(t),(Tw_0)(t))\\ &=\frac{1}{100n^2}\big[(t-\frac{t^3}{n^2})^2+t^2\frac{t^3}{(n+1)^2}+(\frac{3t^2}{(2n)^2})^2+(t-\frac{6t}{n^2})^2\big]\\ &\quad+\frac{t}{800n^3}(t-\int_0^t e^{-ts}\frac{s^3}{n^2}ds)^2\\ &\leq \frac{1}{100n^2}(t^2+\frac{t^5}{(n+1)^2}+\frac{9t^4}{4n^2}+t^2)+\frac{t^3}{800n^3}\leq\frac{6}{n^2}.\end{aligned}$$

When $0<t\leq 2$, we have

$$\begin{aligned}&f_n(t,w_0(t),w_0'(t),w_0''(t),(Tw_0)(t))\\ &=\frac{1}{100n^2}\big[(t-\frac{t^3+t^2+t}{n^2})^2+t^2\frac{t^3+t^2+t}{(n+1)^2}+(\frac{3t^2+2t+1}{(2n)^2})^2+(t-\frac{6t+2}{n^2})^2\big]\\ &\quad+\frac{t}{800n^3}(t-\int_0^t e^{-ts}\frac{s^3+s^2+s}{n^2}ds)^2\\ &\leq \frac{1}{100n^2}(t^2+\frac{t^5+t^4+t^3}{(n+1)^2}+\frac{(t^2+2t+1)^2}{4n^2}+t^2)+\frac{t^3}{800n^3}\leq\frac{6}{n^2}.\end{aligned}$$

Hence v_0,w_0 satisfy (H1). On the other hand, for $t\in J$

$$\begin{gathered}v_0(t)\leq \overline{u}\leq u\leq w_0(t),\quad v_0'(t)\leq\overline{v}\leq v\leq w_0'(t),\\ v_0''(t)\leq\overline{w}\leq w\leq w_0'(t),\quad (Tv_0)(t)\leq\overline{z}\leq u\leq (Tw_0)(t),\end{gathered}$$

we have

$$\begin{gathered}0\leq\overline{u}_n\leq u_n\leq\frac{t^3+t^2+t}{n^2},\quad 0\leq\overline{v}_n\leq v_n\leq\frac{3t^2+2t+1}{n^2},\\ 0\leq\overline{w}_n\leq w_n\leq\frac{6t+2}{n^2},\quad 0\leq\ \overline{z}_n\leq z_n\leq\frac{3t^4+4t^3+6t^2}{12},\end{gathered}$$

$n=1,2,\dots$. Therefore, by (4.2),

$$\begin{aligned}&f_n(t,u,v,w,z)-f_n(t,\overline{u},\overline{v},\overline{w},\overline{z})\\ &\geq\frac{1}{100n^2}\big[(t-u_n)^2-(t-\overline{u}_n)^2+(t-w_n)^2-(t-\overline{w}_n)^2\big]\\ &\quad+\frac{t}{800n^3}\big[(t-z_n)^2-(t-\overline{z}_n)^2\big]\\ &\geq-\frac{1}{100n^2}[2t(u_n-\overline{u}_n)+2t(w_n-\overline{w}_n]-\frac{2t^2}{800n^3}(z_n-\ \overline{z}_n)-\frac{1}{25}(u_n-\ \overline{u}_n)\\ &\quad-\frac{1}{25}(w_n-\overline{w}_n)-\frac{1}{100}(z_n-\overline{z}_n).\end{aligned}$$

Consequently, (H2) is satisfied for $M_0 = 1/25 = M_2$, $M_1 = 0$, $N = 1/100$. It is clear that $k_0 = 1$ and $\tau = 1$, and it is easy to verify that inequalities (2.2) and (2.15) hold. Hence, our conclusion follows from Theorem 3.1. □

References

[1] Guo Dajun and Lakshmikantham.V., *Nonlinear problems in abstract cones*, Academic Press, New York, (1998).

[2] Guo Dajun, *Initial value problems for second impulsive integro-differential equations in Banach spaces*. Chin. Ann. Of Math. 18B:4(1997), 439-448.

[3] Guo Dajun, *Integro-differential equations on unbounded domains in Banach spaces*, Chin. Ann. Of Math. 20B:4(1999), 435-446.

[4] Du, Y., *Fixed points of increasing operators in ordered Banach spaces and applications*, Appl. Anal. 38(1990), 1-20.

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