SELF-ADJOINTNESS OF SCHRÖDINGER-TYPE OPERATORS WITH SINGULAR POTENTIALS ON MANIFOLDS OF BOUNDED GEOMETRY

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Abstract. We consider the Schrödinger type differential expression

\[ H_V = \nabla^* \nabla + V, \]

where \( \nabla \) is a \( C^\infty \)-bounded Hermitian connection on a Hermitian vector bundle \( E \) of bounded geometry over a manifold of bounded geometry \( (M, g) \) with metric \( g \) and positive \( C^\infty \)-bounded measure \( d\mu \), and \( V = V_1 + V_2 \), where \( 0 \leq V_1 \in L^1_{\text{loc}}(\text{End } E) \) and \( 0 \geq V_2 \in L^1_{\text{loc}}(\text{End } E) \) are linear self-adjoint bundle endomorphisms. We give a sufficient condition for self-adjointness of the operator \( S \) in \( L^2(E) \) defined by \( Su = H_V u \) for all \( u \in \text{Dom}(S) = \{ u \in W^{1,2}(E) : \int \langle V_1 u, u \rangle d\mu < +\infty \text{ and } H_V u \in L^2(E) \} \). The proof follows the scheme of T. Kato, but it requires the use of more general version of Kato’s inequality for Bochner Laplacian operator as well as a result on the positivity of \( u \in L^2(M) \) satisfying the equation \( (\Delta_M + b)u = \nu \), where \( \Delta_M \) is the scalar Laplacian on \( M \), \( b > 0 \) is a constant and \( \nu \geq 0 \) is a positive distribution on \( M \).

1. Introduction and main result

Let \( (M, g) \) be a \( C^\infty \) Riemannian manifold without boundary, with metric \( g \), \( \dim M = n \). We will assume that \( M \) is connected. We will also assume that \( M \) has bounded geometry. Moreover, we will assume that we are given a positive \( C^\infty \)-bounded measure \( d\mu \), i.e. in any local coordinates \( x^1, x^2, \ldots, x^n \) there exists a strictly positive \( C^\infty \)-bounded density \( \rho(x) \) such that \( d\mu = \rho(x) dx^1 dx^2 \ldots dx^n \).

Let \( E \) be a Hermitian vector bundle over \( M \). We will assume that \( E \) is a bundle of bounded geometry (i.e. it is supplied by an additional structure: trivializations of \( E \) on every canonical coordinate neighborhood \( U \) such that the corresponding matrix transition functions \( h_{U,U'} \) on all intersections \( U \cap U' \) of such neighborhoods are \( C^\infty \)-bounded, i.e. all derivatives \( \partial^\alpha h_{U,U'}(y) \), where \( \alpha \) is a multiindex, with respect to canonical coordinates are bounded with bounds \( C_\alpha \) which do not depend on the chosen pair \( U, U' \)).
We denote by $L^2(E)$ the Hilbert space of square integrable sections of $E$ with respect to the scalar product

$$\langle u, v \rangle = \int_M \langle u(x), v(x) \rangle \, d\mu(x).$$

(1.1)

Here $\langle \cdot, \cdot \rangle$ denotes the fiberwise inner product in $E_x$.

In what follows, $C^\infty(E)$ denotes smooth sections of $E$, and $C^\infty_c(E)$ denotes smooth compactly supported sections of $E$. Let

$$\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$$

be a Hermitian connection on $E$ which is $C^\infty$-bounded as a linear differential operator, i.e. in any canonical coordinate system $U$ (with the chosen trivializations of $E|_U$ and $(T^*M \otimes E)|_U$), $\nabla$ is written in the form

$$\nabla = \sum_{|\alpha| \leq 1} a_\alpha(y) \partial^\alpha_y,$$

where $\alpha$ is a multiindex, and the coefficients $a_\alpha(y)$ are matrix functions whose derivatives $\partial^\beta_y a_\alpha(y)$ for any multiindex $\beta$ are bounded by a constant $C_\beta$ which does not depend on the chosen canonical neighborhood.

We will consider a Schrödinger type differential expression of the form

$$H_V = \nabla^* \nabla + V,$$

where $V$ is a linear self-adjoint bundle map $V \in L^1_{\text{loc}}(\text{End } E)$. Here

$$\nabla^* : C^\infty(T^*M \otimes E) \to C^\infty(E)$$

is a differential operator which is formally adjoint to $\nabla$ with respect to the scalar product (1.1).

If we take $\nabla = d$, where $d : C^\infty(M) \to \Omega^1(M)$ is the standard differential, then $d^*d : C^\infty(M) \to C^\infty(M)$ is called the scalar Laplacian and will be denoted by $\Delta_M$.

We make the following assumption on $V$.

(A1) $V = V_1 + V_2$, where $0 \leq V_1 \in L^1_{\text{loc}}(\text{End } E)$ and $0 \geq V_2 \in L^1_{\text{loc}}(\text{End } E)$ are linear self-adjoint bundle maps (here the inequalities are understood in the sense of operators $E_x \to E_x$).

By $W^{1,2}(E)$ we denote the completion of the space $C^\infty_c(E)$ with respect to the norm $\| \cdot \|_1$ defined by the scalar product

$$\langle u, v \rangle_1 := \langle u, v \rangle + \langle \nabla u, \nabla v \rangle \quad u, v \in C^\infty_c(E).$$

By $W^{-1,2}(E)$ we will denote the dual of $W^{1,2}(E)$.

2. Quadratic forms

In what follows, all quadratic forms are considered in the Hilbert space $L^2(E)$.

By $h_0$ we denote the quadratic form

$$h_0(u) = \int |\nabla u|^2 \, d\mu$$

with the domain $\text{D}(h_0) = W^{1,2}(E) \subset L^2(E)$. Clearly, $h_0$ is a non-negative, densely defined and closed form.

By $h_1$ we denote the quadratic form

$$h_1(u) = \int \langle V_1 u, u \rangle \, d\mu$$

(2.2)
with the domain
\[ D(h_1) = \{ u \in L^2(E) : \int \langle V_1 u, u \rangle \, d\mu < +\infty \}. \]  (2.3)

Clearly, \( h_1 \) is a non-negative, densely defined, and closed form.

By \( h_2 \) we denote the quadratic form
\[ h_2(u) = \int \langle V_2 u, u \rangle \, d\mu \]  (2.4)

with the domain
\[ D(h_2) = \{ u \in L^2(E) : \int |\langle V_2 u, u \rangle| \, d\mu < +\infty \}. \]  (2.5)

Clearly, \( h_2 \) is a densely defined form. Moreover, \( h_2 \) is symmetric (but not semi-bounded below).

We make the following assumption on \( h_2 \).

A2) Assume that \( h_2 \) is \( h_0 \)-bounded with relative bound \( b < 1 \), i.e.
(i) \( D(h_2) \supset D(h_0) \)
(ii) There exist constants \( a \geq 0 \) and \( 0 \leq b < 1 \) such that
\[ |h_2(u)| \leq a\|u\|^2 + b|h_0(u)|, \quad \text{for all } u \in D(h_0), \]  (2.6)

where \( \| \cdot \| \) denotes the norm in \( L^2(E) \).

Remark 2.1. With the above assumptions on \((M, g), bundle E and connection } \nabla \), Assumption (A2) holds if \( V_2 \in L^p(\operatorname{End} E) \), where \( p = n/2 \) for \( n \geq 3 \), \( p > 1 \) for \( n = 2 \), and \( p = 1 \) for \( n = 1 \). The proof is given in the last section of this article.

As a realization of \( H_V \) in \( L^2(E) \), we define the operator \( S \) in \( L^2(E) \) by the formula
\[ Su = H_V u \] on the domain
\[ \operatorname{Dom}(S) = \{ u \in W^{1,2}(E) : \int \langle V_1 u, u \rangle \, d\mu < +\infty \text{ and } H_V u \in L^2(E) \}. \]  (2.7)

Remark 2.2. For all \( u \in D(h_0) = W^{1,2}(E) \) we have \( \nabla^* \nabla u \in W^{-1,2}(E) \), and from Corollary 3.7 below it follows that for all \( u \in W^{1,2}(E) \cap D(h_1) \), we have \( V u \in L^1_{\text{loc}}(E) \). Thus \( H_V u \) in (2.7) is a distributional section of \( E \), and the condition \( H_V u \in L^2(E) \) makes sense.

We now state the main result.

Theorem 2.3. Assume that \((M, g)\) is a manifold of bounded geometry with positive \( C^\infty \)-bounded measure \( d\mu \), \( E \) is a Hermitian vector bundle of bounded geometry over \( M \), and \( \nabla \) is a \( C^\infty \)-bounded Hermitian connection on \( E \). Suppose that Assumptions (A1) and (A2) hold. Then \( S \) is a semi-bounded below self-adjoint operator.

Remark 2.4. Theorem 2.3 extends a result of T. Kato, cf. Theorem VI.4.6(a) in [8] (see also remark 5(b) in [7]) which was proven for the operator \(-\Delta + V\), where \( \Delta \) is the standard Laplacian on \( \mathbb{R}^n \) with the standard metric and measure, and \( V = V_1 + V_2 \), where \( 0 \leq V_1 \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( 0 \geq V_2 \in L^1_{\text{loc}}(\mathbb{R}^n) \) are as in Assumption (A1) above, and the quadratic form \( h_2 \) corresponding to \( V_2 \) is as in Assumption (A2) above.
3. Proof of Theorem 2.3

We adopt the arguments from Sec. VI.4 in [8] to our setting with the help of more general version of Kato’s inequality (3.1).

We begin with the following variant of Kato’s inequality for Bochner Laplacian (for the proof see Theorem 5.7 in [2]). The original version of Kato’s inequality was proven in Kato [5].

Lemma 3.1. Assume that \((M, g)\) is a Riemannian manifold. Assume that \(E\) is a Hermitian vector bundle over \(M\) and \(\nabla\) is a Hermitian connection on \(E\). Assume that \(w \in L^1_{\text{loc}}(E)\) and \(\nabla^* \nabla w \in L^1_{\text{loc}}(E)\). Then

\[
\Delta_M |w| \leq \text{Re} \langle \nabla^* \nabla w, \text{sign } w \rangle, \tag{3.1}
\]

where

\[
\text{sign } w(x) = \begin{cases} 
\frac{w(x)}{|w(x)|} & \text{if } w(x) \neq 0, \\
0 & \text{otherwise}.
\end{cases}
\]

In what follows, we will use the following Lemma whose proof is given in Appendix B of [2].

Lemma 3.2. Assume that \((M, g)\) is a manifold of bounded geometry with a smooth positive measure \(d\mu\). Assume that \((b + \Delta_M) u = \nu \geq 0, u \in L^2(M)\), where \(b > 0, \Delta_M = d^*d\) is the scalar Laplacian on \(M\), and the inequality \(\nu \geq 0\) means that \(\nu\) is a positive distribution on \(M\), i.e. \(\langle \nu, \phi \rangle \geq 0\) for any \(0 \leq \phi \in C^\infty_c(M)\). Then \(u \geq 0\) (almost everywhere or, equivalently, as a distribution).

Remark 3.3. It is not known whether Lemma 3.2 holds if \(M\) is an arbitrary complete Riemannian manifold. For more details about difficulties in the case of arbitrary complete Riemannian manifolds, see Appendix B of [2].

Lemma 3.4. The quadratic form \(h := (h_0 + h_1) + h_2\) is densely-defined, semi-bounded below and closed.

Proof. Since \(h_0\) and \(h_1\) are non-negative and closed, it follows by Theorem VI.1.31 from [8] that \(h_0 + h_1\) is non-negative and closed. Since \(h_1\) is non-negative, it follows immediately from Assumption (A2) that \(h_2\) is \((b_0 + h_1)\)-bounded with relative bound \(b < 1\). Since \(h_0 + h_1\) is a closed, non-negative form, by Theorem VI.1.33 from [8], it follows that \(h = (h_0 + h_1) + h_2\) is a closed semi-bounded below form. Since \(C^\infty_c(E) \subset D(h_0) \cap D(h_1) \subset D(h_2)\), it follows that \(h\) is densely defined. \(\square\)

In what follows, \(h(\cdot, \cdot)\) will denote the corresponding sesquilinear form obtained from \(h\) via polarization identity.

Self-adjoint operator \(H\) associated to \(h\). Since \(h\) is densely defined, closed and semi-bounded below form in \(L^2(E)\), by Theorem VI.2.1 from [8] there exists a semi-bounded below self-adjoint operator \(H\) in \(L^2(E)\) such that

(i) \(\text{Dom}(H) \subset D(h)\) and

\[
h(u, v) = \langle Hu, v \rangle \quad \text{for all } u \in \text{Dom}(H), \text{ and } v \in D(h).
\]

(ii) \(\text{Dom}(H)\) is a core of \(h\).
(iii) If \( u \in D(h) \), \( w \in L^2(E) \) and \( h(u, v) = (w, v) \) holds for every \( v \) belonging to a core of \( h \), then \( u \in \text{Dom}(H) \) and \( Hu = w \). The semi-bounded below self-adjoint operator \( H \) is uniquely determined by the condition (i).

In what follows we will use the following well-known Lemma.

**Lemma 3.5.** Assume that \( 0 \leq T \in L^1_{\text{loc}}(\text{End} E) \) is a linear self-adjoint bundle map. Assume also that \( u \in Q(T) \), where \( Q(T) = \{ u \in L^2(E) : \langle Tu, u \rangle \in L^1(M) \} \). Then \( Tu \in L^1_{\text{loc}}(E) \).

**Proof.** By adding a constant we can assume that \( T \geq 1 \) (in the operator sense). Assume that \( u \in Q(T) \). We choose (in a measurable way) an orthogonal basis in each fiber \( E_x \) and diagonalize \( 1 \leq T(x) \in \text{End}(E_x) \) to get
\[
T(x) = \text{diag}(c_1(x), c_2(x), \ldots, c_m(x)),
\]
where \( 0 < c_j \in L^1_{\text{loc}}(M) \), \( j = 1, 2, \ldots, m \) and \( m = \dim E_x \).

Let \( u_j(x) (j = 1, 2, \ldots, m) \) be the components of \( u(x) \in E_x \) with respect to the chosen orthogonal basis of \( E_x \). Then for all \( x \in M \)
\[
\langle Tu, u \rangle = \sum_{j=1}^{m} c_j(x)|u_j(x)|^2.
\]
Since \( u \in Q(T) \), we know that \( 0 < \int \langle Tu, u \rangle \, d\mu < +\infty \). Since \( c_j > 0 \), it follows that \( c_j|u_j|^2 \in L^1_1(M) \), for all \( j = 1, 2, \ldots, m \).

Now, for all \( x \in M \) and \( j = 1, 2, \ldots, m \)
\[
2|c_j u_j| = 2|c_j||u_j| \leq |c_j| + |c_j||u_j|^2,
\]
The right hand side of (3.2) is clearly in \( L^1_{\text{loc}}(M) \). Therefore \( c_j u_j \in L^1_{\text{loc}}(M) \). But \( \langle Tu \rangle(x) \) has components \( c_j(x)u_j(x) (j = 1, 2, \ldots, m) \) with respect to chosen bases of \( E_x \). Therefore \( Tu \in L^1_{\text{loc}}(E) \), and the Lemma is proven. \( \square \)

The following corollary follows immediately from Lemma 3.5.

**Corollary 3.6.** If \( u \in D(h_1) \), then \( V_1 u \in L^1_{\text{loc}}(E) \).

**Corollary 3.7.** If \( u \in D(h) \), then \( Vu \in L^1_{\text{loc}}(E) \).

**Proof.** Let \( u \in D(h) = D(h_0) \cap D(h_1) \). By Assumption (A1) we have \( V = V_1 + V_2 \), where \( 0 \leq V_1 \in L^1_{\text{loc}}(\text{End} E) \) and \( 0 \geq V_2 \in L^1_{\text{loc}}(\text{End} E) \). By Corollary 3.6 it follows that \( V_1 u \in L^1_{\text{loc}}(E) \) and since \( D(h) \subset D(h_2) \), by Lemma 3.5 we have \( -V_2 u \in L^1_{\text{loc}}(E) \). Thus \( V u \in L^1_{\text{loc}}(E) \), and the corollary is proven. \( \square \)

**Lemma 3.8.** The following operator relation holds: \( H \subset S \).

**Proof.** We will show that for all \( u \in \text{Dom}(H) \), we have \( Hu = H_S u \). Let \( u \in \text{Dom}(H) \). By property (i) of operator \( H \) we have \( u \in D(h) \), hence by Corollary 3.7 we get \( Vu \in L^1_{\text{loc}}(E) \). Then, for any \( v \in C_c^\infty(E) \), we have
\[
(Hu, v) = h(u, v) = (\nabla u, \nabla v) + \int (Vu, v) \, d\mu,
\]
where \( (\cdot, \cdot) \) denotes the \( L^2 \)-inner product.

The first equality in (3.3) holds by property (i) of operator \( H \), and the second equality holds by definition of \( h \).
Hence, using integration by parts in the first term on the right hand side of the second equality in (3.3) (see, for example, Lemma 8.8 from [2]), we get
\[(u, \nabla^* \nabla v) = \int \langle Hu - Vu, v \rangle \, d\mu, \quad \text{for all} \quad v \in C^\infty_c(E). \tag{3.4}\]
Since \(V u \in L^1_{\text{loc}}(E)\) and \(H u \in L^2(E)\), it follows that \((Hu - Vu) \in L^1_{\text{loc}}(E)\), and (3.4) implies \(\nabla^* \nabla u = Hu - Vu\) (as distributional sections of \(E\)). Therefore,
\[\nabla^* \nabla u + Vu = Hu,\]
and this shows that \(Hu = Hv u\) for all \(u \in \text{Dom}(H)\).

Now by definition of \(S\) it follows that \(\text{Dom}(H) \subset \text{Dom}(S)\) and \(Hu = Su\) for all \(u \in \text{Dom}(H)\). Therefore \(H \subset S\), and the Lemma is proven. \(\square\)

**Lemma 3.9.** \(C^\infty_c(E)\) is a core of the quadratic form \(h_0 + h_1\).

**Proof.** We need to show that \(C^\infty_c(E)\) is dense in the Hilbert space \(D(h_0 + h_1) = D(h_0) \cap D(h_1)\) with the inner product
\[(u, v)_{h_0 + h_1} := h_0(u, v) + h_1(u, v) + (u, v),\]
where \((\cdot, \cdot)\) is the inner product in \(L^2(E)\).

Let \(u \in D(h_0 + h_1)\) and \((u, v)_{h_0 + h_1} = 0\) for all \(v \in C^\infty_c(E)\). We will show that \(u = 0\). We have
\[0 = h_0(u, v) + h_1(u, v) + (u, v) = (u, \nabla^* \nabla v) + \int \langle V_1 u, v \rangle \, d\mu + (u, v). \tag{3.5}\]
Here we used integration by parts in the first term on the right hand side of the second equality.

By Corollary 3.6 it follows that \(V_1 u \in L^1_{\text{loc}}(E)\), and from (3.5) we get the following equality of distributional sections of \(E\):
\[\nabla^* \nabla u = -(V_1 - 1)u. \tag{3.6}\]
From (3.6) we have \(\nabla^* \nabla u \in L^1_{\text{loc}}(E)\). By Lemma 3.1 and by (3.6), we obtain
\[\Delta_M |u| \leq \text{Re} \langle \nabla^* \nabla u, \text{sign } u \rangle = \langle -(V_1 - 1)u, \text{sign } u \rangle \leq |u| \tag{3.7}\]
The last inequality in (3.7) follows since \(V_1 \geq 0\) (as an operator \(E_x \rightarrow E_x\)). Therefore,
\[(\Delta_M + 1)|u| \leq 0. \tag{3.8}\]
By Lemma 3.2, it follows that \(|u| \leq 0\). So \(u = 0\), and the proof is complete. \(\square\)

**Lemma 3.10.** \(C^\infty_c(E)\) is a core of the quadratic form \(h = (h_0 + h_1) + h_2\).

Since the quadratic form \(h_2\) is \((h_0 + h_1)\)-bounded, the lemma follows immediately from Lemma 3.9.

**Proof of Theorem 2.3.** We will show that \(S = H\). By Lemma 3.8 we have \(H \subset S\), so it is enough to show that \(\text{Dom}(S) \subset \text{Dom}(H)\).

Let \(u \in \text{Dom}(S)\). By definition of \(\text{Dom}(S)\), we have \(u \in D(h_0) \subset D(h_2)\) and \(u \in D(h_1)\). Hence \(u \in D(h)\). For all \(v \in C^\infty_c(E)\) we have
\[h(u, v) = h_0(u, v) + h_1(u, v) + h_2(u, v) = (u, \nabla^* \nabla v) + \int \langle V_1 u, v \rangle \, d\mu = (Hu, v).\]
The last equality holds since \(H V_1 u = Su \in L^2(E)\). By Lemma 3.10 it follows that \(C^\infty_c(E)\) is a form core of \(h\). Now from property (iii) of operator \(H\) we have \(u \in \text{Dom}(H)\) with \(Hu = Hv u\). This concludes the proof of the Theorem. \(\square\)
Proof of Remark 2.1. Let $p$ be as in Remark 2.1. We may assume that $\|V_2\|_{L^p(\text{End } E)}$ is arbitrarily small because there exists a sequence $V_2^{(k)} \in L^\infty(\text{End } E) \cap L^p(\text{End } E)$, $k \in \mathbb{Z}_+$, such that
$$\|V_2^{(k)} - V_2\|_{L^p(\text{End } E)} \to 0, \text{ as } k \to \infty,$$
and $V_2^{(k)}$, $k \in \mathbb{Z}_+$, contributes to $h_2$ only a bounded form.

For the rest of this article, we will assume that $\|V_2\|_{L^p(\text{End } E)}$ is arbitrarily small. By Cauchy-Schwartz inequality and Hölder’s inequality we have
$$\left|\int \langle V_2 u, u \rangle \, dp\right| \leq \int |V_2 u| \, dp \leq \int |V_2| \, dp \leq \|V_2\|_{L^p(\text{End } E)} \|u\|_{L^2(E)}^2,$$ (3.9)
where $|V_2|$ denotes the norm of the operator $V_2(x): E_x \to E_x$ and
$$\frac{1}{p} + \frac{2}{t} = 1. \quad (3.10)$$

With our assumptions on $(M, g)$, $E$ and $\nabla$, the usual Sobolev embedding theorems for $W^{1,2}(\mathbb{R}^n)$ also hold for $W^{1,2}(E)$ (see Sec. A1.1 in [10]).

For $n \geq 3$, we know by hypothesis that $p = n/2$, so from (3.10) we get $1/t = 1/2 - 1/n$. By the Sobolev embedding theorem (see, for example, the first part of Theorem 2.10 in [1]) we have
$$\|u\|_{L^t(E)} \leq C(\|\nabla u\|_{L^2(\text{End } E)} + \|u\|_{L^2(E)}), \quad \text{for all } u \in W^{1,2}(E),$$
where $C > 0$ is a positive constant.

For $n = 2$, we know by hypothesis that $p > 1$, so from (3.10) we get $2 < t < \infty$. In this case, it is well-known (see, e.g., the first part of Theorem 2.10 in [1]) that
$$\|u\|_{L^t(E)} \leq C(\|\nabla u\|_{L^2(\text{End } E)} + \|u\|_{L^2(E)}), \quad \text{for all } u \in W^{1,2}(E),$$
where $C > 0$ is a positive constant.

For $n = 1$, we know by hypothesis that $p = 1$, so from (3.10) we get $t = \infty$. In this case, it is well-known (see e.g. the second part of Theorem 2.10 in [1]) that
$$\|u\|_{L^\infty(E)} \leq C(\|\nabla u\|_{L^2(\text{End } E)} + \|u\|_{L^2(E)}), \quad \text{for all } u \in W^{1,2}(E),$$
where $C > 0$ is a positive constant.

Combining each of the last three inequalities with (3.9), we get (2.6) (with constant $b < 1$ because $\|V_2\|_{L^p(\text{End } E)}$ is arbitrarily small). \qed

References


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