

Order and hyper-order of entire solutions of linear differential equations with entire coefficients *

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Abstract

In this paper, we investigate the growth of solutions of the differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$

where $A_0(z), \dots, A_{k-1}(z)$ are entire functions with $A_0(z) \not\equiv 0$. We will show that if the coefficients satisfy certain growth conditions, then every finite order solution of the equation will satisfy certain other growth conditions. We will also find conditions on the coefficients so that every solution $f \not\equiv 0$ will have infinite order and we estimate in one case the lower bounds of the hyper-order.

1 Introduction and statement of results

We use the standard notations of the Nevanlinna theory [6]. The order of an entire function f is defined as

$$\sigma(f) = \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} = \limsup_{r \rightarrow +\infty} \frac{\log \log M(r, f)}{\log r}, \quad (1.1)$$

where $T(r, f)$ is the Nevanlinna characteristic function of f and $M(r, f) = \max_{|z|=r} |f(z)|$ (see [6]). To express the rate of growth of entire function of infinite order, we recall the following concept.

Definition The hyper-order [1, 7] of an entire function f is defined as

$$\sigma_2(f) = \limsup_{r \rightarrow \infty} \frac{\log \log T(r, f)}{\log r} = \limsup_{r \rightarrow \infty} \frac{\log \log \log M(r, f)}{\log r}. \quad (1.2)$$

We define the linear measure of a set $E \subset [0, +\infty)$ by $m(E) = \int_0^{+\infty} \chi_E(t) dt$ where χ_E is the characteristic function of E . The upper and the lower densities of E are defined by

$$\overline{\text{dens}}E = \limsup_{r \rightarrow +\infty} \frac{m(E \cap [0, r])}{r}, \quad \underline{\text{dens}}E = \liminf_{r \rightarrow +\infty} \frac{m(E \cap [0, r])}{r}. \quad (1.3)$$

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For $k \geq 2$, we consider the linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \quad (1.4)$$

where $A_0(z), \dots, A_{k-1}(z)$ are entire functions with $A_0(z) \not\equiv 0$. It is well-known that all solutions of (1.4) are entire functions, and if some coefficients of (1.4) are transcendental, then (1.4) has at least one solution with order $\sigma(f) = +\infty$. A question arises:

What conditions on $A_0(z), \dots, A_{k-1}(z)$ will guarantee that every solution $f \not\equiv 0$, of (1.4) has infinite order?

According to [5], [9, pp. 106-108] and [10, pp. 65-67], if $A_0(z), \dots, A_{k-1}(z)$ are polynomials with $A_0(z) \not\equiv 0$, then every solution of (1.4) is an entire function with finite rational order.

In this paper we prove results concerning the above question and other results about solutions of finite order and their properties.

In the study of the differential equation

$$f'' + A(z)f' + B(z)f = 0, \quad (1.5)$$

where $A(z)$ and $B(z) \not\equiv 0$ are entire functions, Gundersen proved the following result.

Theorem 1.1 ([3, p. 417]) *Let $A(z)$ and $B(z) \not\equiv 0$ be entire functions such that for real constants $\alpha, \beta, \theta_1, \theta_2$ satisfy $\alpha > 0, \beta > 0$, and $\theta_1 < \theta_2$, we have*

$$|A(z)| \geq \exp\{(1 + o(1))\alpha|z|^\beta\} \quad (1.6)$$

and

$$|B(z)| \leq \exp\{o(1)|z|^\beta\} \quad (1.7)$$

as $z \rightarrow \infty$ in $\theta_1 \leq \arg z \leq \theta_2$. Let $\varepsilon > 0$ be a given small constant, and let $S(\varepsilon)$ denote the angle $\theta_1 + \varepsilon \leq \arg z \leq \theta_2 - \varepsilon$. If $f \not\equiv 0$ is a solution of (1.5), where $\sigma(f) < +\infty$, then the following conditions hold:

- (i) *There exists a constant $b \neq 0$ such that $f(z) \rightarrow b$ as $z \rightarrow \infty$ in $S(\varepsilon)$. Furthermore, as $z \rightarrow \infty$ in $S(\varepsilon)$,*

$$|f(z) - b| \leq \exp\{-(1 + o(1))\alpha|z|^\beta\} \quad (1.8)$$

- (ii) *For each integer $k \geq 1$, as $z \rightarrow \infty$ in $S(\varepsilon)$,*

$$|f^{(k)}(z)| \leq \exp\{-(1 + o(1))\alpha|z|^\beta\}. \quad (1.9)$$

In the same paper, Gundersen proved the following statement.

Theorem 1.2 ([3, p. 418]) *Let $A(z)$ and $B(z) \not\equiv 0$ be entire functions, and let $\alpha > 0$, $\beta > 0$ be constants, with $\sigma(B) < \beta$. Suppose that for any given $\varepsilon > 0$, there exist two finite collections of real numbers $\{\phi_k\}$ and $\{\theta_k\}$ that satisfy $\phi_1 < \theta_1 < \phi_2 < \theta_2 < \dots < \phi_n < \theta_n < \phi_{n+1}$, where $\phi_{n+1} = \phi_1 + 2\pi$ and*

$$\sum_{k=1}^n (\phi_{k+1} - \theta_k) < \varepsilon, \quad (1.10)$$

such that

$$|A(z)| \geq \exp\{(1 + o(1))\alpha|z|^\beta\} \quad (1.11)$$

as $z \rightarrow \infty$ in $\phi_k \leq \arg z \leq \theta_k$ ($k = 1, 2, \dots, n$). Then every solution $f \not\equiv 0$ of (1.5) is of infinite order.

Recently, for the second order equation (1.5), Ki-Ho Kwon obtained in [7] the following results:

Theorem 1.3 ([7, p. 488]) *Let $A(z)$ and $B(z)$ be entire functions satisfying $\sigma(B) < \sigma(A)$. Then every solution $f \not\equiv 0$ of finite order of (1.5) satisfies $\sigma(f) \geq \sigma(A)$.*

Theorem 1.4 ([7, p. 489]) *Let $A(z)$ and $B(z)$ be entire functions where $0 < \sigma(B) < 1/2$, and let there exist a real constant $\beta < \sigma(B)$ and a set $E_\beta \subset [0, +\infty)$ with $\underline{\text{dens}}E_\beta = 1$ such that for all $r \in E_\beta$, we have*

$$\min_{|z|=r} |A(z)| \leq \exp(r^\beta). \quad (1.12)$$

Then every solution $f \not\equiv 0$ of (1.5) is of infinite order with

$$\sigma_2(f) = \limsup_{r \rightarrow \infty} \frac{\log \log T(r, f)}{\log r} \geq \sigma(B).$$

We shall investigate generalizations of problems of the above type to higher order homogeneous linear differential equations.

Theorem 1.5 *Let $A_0(z), \dots, A_{k-1}(z)$ with $A_0(z) \not\equiv 0$ be entire functions such that for real constants $\alpha, \beta, \theta_1, \theta_2$, where $\alpha > 0, \beta > 0$, and $\theta_1 < \theta_2$, we have*

$$|A_1(z)| \geq \exp\{(1 + o(1))\alpha|z|^\beta\} \quad (1.13)$$

and

$$|A_j(z)| \leq \exp\{o(1)|z|^\beta\}, \quad j = 0, 2, \dots, k-1 \quad (1.14)$$

as $z \rightarrow \infty$ in $\theta_1 \leq \arg z \leq \theta_2$. Let $\varepsilon > 0$ be a given small constant, and let $S(\varepsilon)$ denote the angle $\theta_1 + \varepsilon \leq \arg z \leq \theta_2 - \varepsilon$.

If $f \not\equiv 0$ is a solution of (1.4) where $\sigma(f) < +\infty$, then the following conditions hold:

(i) There exists a constant $b \neq 0$ such that $f(z) \rightarrow b$ as $z \rightarrow \infty$ in $S(\varepsilon)$.
Furthermore, as $z \rightarrow \infty$ in $S(\varepsilon)$,

$$|f(z) - b| \leq \exp\{-(1 + o(1))\alpha|z|^\beta\}. \quad (1.15)$$

(ii) For each integer $m \geq 1$, as $z \rightarrow \infty$ in $S(\varepsilon)$,

$$|f^{(m)}(z)| \leq \exp\{-(1 + o(1))\alpha|z|^\beta\}. \quad (1.16)$$

Theorem 1.6 Let $A_0(z), \dots, A_{k-1}(z)$ with $A_0(z) \not\equiv 0$ be entire functions, and let $\alpha > 0$, $\beta > 0$ be constants where $\sigma(A_j) < \beta$ ($j = 0, 2, \dots, k-1$). Suppose that for any given $\varepsilon > 0$, there exist two finite collections of real numbers $\{\phi_s\}$ and $\{\theta_s\}$ that satisfy $\phi_1 < \theta_1 < \phi_2 < \theta_2 < \dots < \phi_n < \theta_n < \phi_{n+1}$ where $\phi_{n+1} = \phi_1 + 2\pi$ and

$$\sum_{s=1}^n (\phi_{s+1} - \theta_s) < \varepsilon, \quad (1.17)$$

such that

$$|A_1(z)| \geq \exp\{(1 + o(1))\alpha|z|^\beta\} \quad (1.18)$$

as $z \rightarrow \infty$ in $\phi_s \leq \arg z \leq \theta_s$ ($s = 1, 2, \dots, n$). Then every solution $f \not\equiv 0$ of (1.4) is of infinite order.

Here we estimate the lower bounds for the order of solution $f \not\equiv 0$ of (1.4) when f is of finite order.

Theorem 1.7 Let $A_0(z), \dots, A_{k-1}(z)$ be entire functions that satisfy $\max\{\sigma(A_j) : j = 0, 2, \dots, k-1\} < \sigma(A_1)$. Then every solution $f \not\equiv 0$ of (1.4) of finite order satisfies $\sigma(f) \geq \sigma(A_1)$.

Remark If $\max\{\sigma(A_j) : j = 0, 2, \dots, k-1\} = \sigma(A_1)$, the conclusion of Theorem 1.7 is in general false. Indeed $f(z) = z^2$ satisfies $f'' + z^2 e^z f'' + z e^z f' - 4e^z f = 0$.

Now, we estimate the lower bounds on the hyper-order if every solution $f \not\equiv 0$ is of infinite order.

Theorem 1.8 Let $A_0(z), \dots, A_{k-1}(z)$ be entire functions where $0 < \sigma(A_0) < 1/2$, and let there exist a real constant $\beta < \sigma(A_0)$ and a set $E_\beta \subset [0, +\infty)$ with $\text{dens} E_\beta = 1$ such that for all $r \in E_\beta$, we have

$$\min_{|z|=r} |A_j(z)| \leq \exp(r^\beta) \quad j = 1, 2, \dots, k-1. \quad (1.19)$$

Then every solution $f \not\equiv 0$ of (1.4) is of infinite order with

$$\sigma_2(f) = \limsup_{r \rightarrow \infty} \frac{\log \log T(r, f)}{\log r} \geq \sigma(A_0).$$

2 Preliminary Lemmas

We need the following lemmas for the proofs of our theorems.

Lemma 2.1 ([4, p. 89]) *Let w be a transcendental entire function of finite order σ . Let $\Gamma = \{(k_1, j_1), (k_2, j_2), \dots, (k_m, j_m)\}$ denote a finite set of distinct pairs of integers that satisfy $k_i > j_i \geq 0$, $i = 1, \dots, m$, and let $\varepsilon > 0$ be a given constant. Then there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) - E$, then there is a constant $R_0 = R_0(\psi_0) > 0$ such that for all z satisfying $\arg z = \psi_0$ and $|z| \geq R_0$, and for all $(k, j) \in \Gamma$, we have*

$$\left| \frac{w^{(k)}(z)}{w^{(j)}(z)} \right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}. \quad (2.1)$$

Lemma 2.2 ([3, p. 421]) *Let w be an entire function such that $|w'(z)|$ is unbounded on some ray $\arg z = \theta$. Then there exists an infinite sequence of points $z_n = r_n e^{i\theta}$ where $r_n \rightarrow +\infty$ such that $w'(z_n) \rightarrow \infty$ and*

$$\left| \frac{w(z_n)}{w'(z_n)} \right| \leq (1 + o(1))|z_n|, \quad (2.2)$$

as $z_n \rightarrow \infty$.

Lemma 2.3 ([3, p. 421]) *Let w be analytic on a ray $\arg z = \theta$, and suppose that for some constant $\alpha > 1$, we have*

$$\left| \frac{w'(z)}{w(z)} \right| = O(|z|^{-\alpha}), \quad (2.3)$$

as $z \rightarrow \infty$ along $\arg z = \theta$. Then there exists a constant $c \neq 0$ such that $w(z) \rightarrow c$ as $z \rightarrow \infty$ along $\arg z = \theta$.

Lemma 2.4 ([2]) *Let $f(z)$ be an entire function of order σ where $0 < \sigma < 1/2$, and let $\varepsilon > 0$ be a given constant. Then there exists a set $E_2 \subset [0, +\infty)$ with $\overline{\text{dens}} E_2 \geq 1 - 2\sigma$ such that for all z satisfying $|z| = r \in E_2$, we have*

$$|f(z)| \geq \exp(r^{\sigma-\varepsilon}). \quad (2.4)$$

Lemma 2.5 ([1]) *Let H be a set of complex numbers satisfying $\overline{\text{dens}}\{|z| : z \in H\} > 0$, and let $A_0(z), \dots, A_{k-1}(z)$ be entire functions such that for some constants $0 \leq \beta < \alpha$ and $\mu > 0$, we have*

$$|A_0(z)| \geq \exp(\alpha|z|^\mu) \quad (2.5)$$

and

$$|A_j(z)| \leq \exp(\beta|z|^\mu) \quad j = 1, \dots, k-1 \quad (2.6)$$

as $z \rightarrow \infty$ for $z \in H$. Then every solution $f \not\equiv 0$ of (1.4) satisfies $\sigma(f) = +\infty$ and $\sigma_2(f) \geq \mu$.

3 Proofs of Theorem 1.5

Suppose that $f \neq 0$ is a solution of (1.4) with $\sigma(f) < \infty$. Set $\delta = \sigma(f)$. Then from Lemma 2.1, there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) - E$, then

$$\left| \frac{f^{(j)}(z)}{f'(z)} \right| = o(1)|z|^{(j-1)\delta} \quad j = 2, \dots, k \quad (3.1)$$

as $z \rightarrow \infty$ along $\arg z = \psi_0$.

Now suppose that $|f'(z)|$ is unbounded on some ray $\arg z = \phi_0$ where $\phi_0 \in [\theta_1, \theta_2] - E$. Then from Lemma 2.2, there exists an infinite sequence of points $z_n = r_n e^{i\phi_0}$ where $r_n \rightarrow +\infty$ such that $f'(z_n) \rightarrow \infty$ and

$$\left| \frac{f(z_n)}{f'(z_n)} \right| \leq (1 + o(1))|z_n|. \quad (3.2)$$

From (1.4), we have

$$|A_1(z)| \leq \left| \frac{f^{(k)}}{f'} \right| + |A_{k-1}(z)| \left| \frac{f^{(k-1)}}{f'} \right| + \dots + |A_2(z)| \left| \frac{f''}{f'} \right| + |A_0(z)| \left| \frac{f}{f'} \right|. \quad (3.3)$$

By using (3.2), (3.1), (1.13), and (1.14), we will obtain a contradiction in (3.3) as $z \rightarrow \infty$. Therefore, $|f'(z)|$ is bounded on any ray $\arg z = \phi$ where $\phi \in [\theta_1, \theta_2] - E$. It then follows from the classical Phragmén-Lindelöf theorem [8, p. 214] that there exists a constant $M > 0$ such that

$$|f'(z)| \leq M \quad (3.4)$$

for all $z \in S(\varepsilon)$. If $\theta_0 \in [\theta_1 + \varepsilon, \theta_2 - \varepsilon] - E$, then when $\arg z = \theta_0$, we obtain from (3.4) that

$$|f(z)| \leq |f(0)| + \left| \int_0^z f'(u) du \right| \leq |f(0)| + M|z|. \quad (3.5)$$

From (3.5), (3.1), and (1.4), we obtain

$$\begin{aligned} |A_1(z)||f'(z)| &\leq \left\{ o(1)|z|^{(k-1)\delta} + o(1)|z|^{(k-2)\delta}|A_{k-1}(z)| + \dots \right. \\ &\quad \left. + o(1)|z|^\delta|A_2(z)| \right\} |f'(z)| + |A_0(z)| \{|f(0)| + M|z|\} \end{aligned} \quad (3.6)$$

as $z \rightarrow \infty$ along $\arg z = \theta_0$. From (3.6), (1.13), and (1.14), we can deduce that

$$|f'(z)| \leq \exp\{-(1 + o(1))\alpha|z|^\beta\} \quad (3.7)$$

as $z \rightarrow \infty$ along $\arg z = \theta_0$. By using an application of the Phragmén-Lindelöf theorem to (3.7), it can be deduced that

$$|f'(z)| \leq \exp\{-(1 + o(1))\alpha|z|^\beta\} \quad (3.8)$$

as $z \rightarrow \infty$ in $S(2\varepsilon)$. This gives $m = 1$ in (1.16).

Now let $z \in S(3\varepsilon)$ where $|z| > 1$, let γ be a circle of radius one with center at z , and let $m \geq 1$ be an integer. Then from the Cauchy integral formula and (3.8), we obtain as $z \rightarrow \infty$ in $S(3\varepsilon)$,

$$|f^{(m)}(z)| \leq \frac{(m-1)!}{2\pi} \oint_{\gamma} \frac{|f'(u)|}{|z-u|^m} |du| \leq \exp\{-(1+o(1))\alpha|z|^\beta\}. \quad (3.9)$$

This proves (1.16). Now fix θ , where $\theta_1 + \varepsilon \leq \theta \leq \theta_2 - \varepsilon$, and set

$$a_0 = \int_0^{+\infty} f'(te^{i\theta})e^{i\theta} dt, \quad (3.10)$$

where we note that $a_0 \in \mathbb{C}$ from (1.16). Let $z = |z|e^{i\psi}$ where $\theta_1 + \varepsilon \leq \psi \leq \theta_2 - \varepsilon$. Then from the Cauchy theorem and (3.10), we obtain

$$\begin{aligned} f(z) - f(0) - a_0 &= \int_0^z f'(u)du - \int_0^{+\infty} f'(te^{i\theta})e^{i\theta} dt \\ &= \int_{\theta}^{\psi} f'(|z|e^{i\zeta})i|z|e^{i\zeta}d\zeta - \int_{|z|}^{+\infty} f'(te^{i\theta})e^{i\theta} dt. \end{aligned} \quad (3.11)$$

Since

$$|f'(z)| \leq \exp\{-(1+o(1))\alpha|z|^\beta\}$$

as $z \rightarrow \infty$ in $S(\varepsilon)$, then from (3.11), we get

$$\begin{aligned} &|f(z) - b| \\ &= \left| \int_{\theta}^{\psi} f'(|z|e^{i\zeta})i|z|e^{i\zeta}d\zeta - \int_{|z|}^{+\infty} f'(te^{i\theta})e^{i\theta} dt \right| \\ &\leq |\psi - \theta||z| \exp\{-(1+o(1))\alpha|z|^\beta\} + \int_{|z|}^{+\infty} \exp\{-(1+o(1))\alpha t^\beta\} dt \\ &\leq |\psi - \theta||z| \exp\{-(1+o(1))\alpha|z|^\beta\} \\ &\quad + \frac{1}{(1+o(1))\alpha\beta\frac{|z|^{\beta-1}}{2} \exp\{(1+o(1))\alpha\frac{|z|^\beta}{2}\}} \int_{|z|}^{+\infty} \frac{(1+o(1))\alpha\beta\frac{t^{\beta-1}}{2}}{\exp\{(1+o(1))\alpha\frac{t^\beta}{2}\}} dt \\ &\leq |\psi - \theta||z| \exp\{-(1+o(1))\alpha|z|^\beta\} \\ &\quad + \frac{1}{(1+o(1))\alpha\beta\frac{|z|^{\beta-1}}{2} \exp\{(1+o(1))\alpha\frac{|z|^\beta}{2}\}} \exp\{-(1+o(1))\alpha\frac{|z|^\beta}{2}\} \\ &\leq \exp\{-(1+o(1))\alpha|z|^\beta\} \end{aligned} \quad (3.12)$$

as $z \rightarrow \infty$ in $S(\varepsilon)$, where $b = f(0) + a_0$. Note that a_0 in (3.10) is independent of θ . Since (3.12) is the inequality (1.15), it remains only to show that $b \neq 0$.

From Lemma 2.1, there exists a ray $\arg z = \psi_1$ where $\theta_1 + \varepsilon \leq \psi_1 \leq \theta_2 - \varepsilon$, such that

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| = o(1)|z|^{j\delta} \quad j = 2, \dots, k \quad (3.13)$$

as $z \rightarrow \infty$ along $\arg z = \psi_1$, where $\delta = \sigma(f)$. Then from (3.13), (1.13), (1.14), and (1.4), we have

$$\begin{aligned} \left| \frac{f'(z)}{f(z)} \right| &\leq \left| \frac{A_0(z)}{A_1(z)} \right| + |A_2(z)| \left| \frac{f''(z)}{f(z)A_1(z)} \right| + \dots \\ &\quad + |A_{k-1}(z)| \left| \frac{f^{(k-1)}(z)}{f(z)A_1(z)} \right| + \left| \frac{f^{(k)}(z)}{f(z)A_1(z)} \right| \\ &\leq \exp\{-(1+o(1))\alpha|z|^\beta\} \end{aligned} \quad (3.14)$$

as $z \rightarrow \infty$ along $\arg z = \psi_1$. By applying Lemma 2.3 to (3.14), and noting that $f(z) \rightarrow b$ as $z \rightarrow \infty$ in $S(\varepsilon)$ from (3.12), we see that $b \neq 0$. The proof of Theorem 1.5 is complete. \diamond

The next example illustrates Theorem 1.5.

Example 3.1 Consider the differential equation

$$f''' + e^z f'' - 2e^{-z} f' + (1 - e^z)f = 0. \quad (3.15)$$

In this equation, for $z = re^{i\theta}$ ($r \rightarrow +\infty$) and $\frac{2\pi}{3} \leq \theta \leq \frac{3\pi}{4}$, we have

$$\begin{aligned} |A_1(z)| &= |-2e^{-z}| = 2 \exp(-r \cos \theta) \geq \exp((1+o(1))\frac{r}{2}), \\ |A_0(z)| &= |1 - e^z| \leq 1 + \exp(r \cos \theta) \leq \exp(o(1)r) \\ |A_2(z)| &= |e^z| = \exp(r \cos \theta) \leq \exp(o(1)r). \end{aligned}$$

It is easy to see that the conditions (1.13) and (1.14) of Theorem 1.5 are satisfied ($\alpha = \frac{1}{2}$, $\beta = 1$). The function $f(z) = e^z + 2$ with $\sigma(f) = 1$ satisfies equation (3.15) and the relations (1.15), (1.16).

4 Proof of Theorem 1.6

Suppose that $f \not\equiv 0$ is a solution of (1.4) where $\sigma(f) < \infty$. Let $\varepsilon > 0$, $\{\phi_s\}$ and $\{\theta_s\}$ be as in the hypothesis. From (1.18) and $\sigma(A_j) < \beta$ ($j = 0, 2, \dots, k-1$), it follows from Theorem 1.5 (i) that $|f(z)|$ is bounded within each angle $\phi_s + \varepsilon \leq \arg z \leq \theta_s - \varepsilon$ ($s = 1, 2, \dots, n$). Since ε can be arbitrarily small, it follows from (1.17) and the Phragmén-Lindelöf theorem that $|f(z)|$ is bounded in the whole finite plane. Thus f is a nonzero constant from Liouville's theorem, and this contradicts (1.4).

The next two examples illustrate Theorem 1.6.

Example 4.1 Consider the differential equation

$$f''' + \cos \sqrt{z} f'' + \sin z f' + \frac{\sin \sqrt{z}}{\sqrt{z}} f = 0. \quad (4.1)$$

Then from Theorem 1.6, it follows that every solution $f \not\equiv 0$ is of infinite order.

Example 4.2 Suppose $A_0(z), \dots, A_{k-1}(z)$ with $A_0(z) \not\equiv 0$ be entire functions and $n \geq 1$ an integer. Then from Theorem 1.6, it follows that every solution $f \not\equiv 0$ of following two equations is of infinite order:

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_2(z)f'' + \sin(z^n)f' + A_0(z)f = 0, \quad (4.2)$$

where $\rho(A_j) < n$, $j = 0, 2, 3, \dots, k-1$; and

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_2(z)f'' + \cos(z^{n/2})f' + A_0(z)f = 0, \quad (4.3)$$

where $\rho(A_j) < n/2$, $j = 0, 2, 3, \dots, k-1$.

5 Proof of Theorem 1.7

Set $\max\{\sigma(A_j) : j = 0, 2, \dots, k-1\} = \beta < \alpha = \sigma(A_1)$. Suppose that $f \not\equiv 0$ is a solution of (1.4) with $\sigma(f) < +\infty$. It follows from (1.4), that

$$-A_1(z) = \frac{f^{(k)}}{f'} + A_{k-1}(z)\frac{f^{(k-1)}}{f'} + \dots + A_2(z)\frac{f''}{f'} + A_0(z)\frac{f}{f'}. \quad (5.1)$$

Hence from Nevanlinna's fundamental results on meromorphic functions [6, p. 5], [6, Theorem 2.2, p. 34] and [6, Theorem 3.1, p. 55], we have

$$\begin{aligned} m(r, A_1) &\leq m(r, A_0) + m(r, A_2) + \dots + m(r, A_{k-1}) \\ &\quad + \sum_{j=2}^k m(r, \frac{f^{(j)}}{f'}) + m(r, \frac{f}{f'}) + O(1) \\ &\leq m(r, A_0) + m(r, A_2) + \dots + m(r, A_{k-1}) + m(r, \frac{f}{f'}) + O(\log r) \end{aligned} \quad (5.2)$$

holds for all r outside a set $E \subset (0, +\infty)$ with a linear measure $m(E) = \delta < +\infty$. Here the notation $m(r, h)$ for a meromorphic function h is defined by

$$m(r, h) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |h(re^{i\theta})| d\theta, \quad (5.3)$$

which is equal to $T(r, h)$ if h is entire. It follows from (5.2) that for $r \notin E$

$$T(r, A_1) \leq T(r, A_0) + T(r, A_2) + \dots + T(r, A_{k-1}) + 2T(r, f) + O(\log r), \quad (5.4)$$

since $m(r, f/f') \leq 2T(r, f) + O(1)$, $r \notin E$. Since $\sigma(A_1) = \alpha$, there exists $\{r'_n\}$ with $r'_n \rightarrow \infty$, such that

$$\lim_{r'_n \rightarrow \infty} \frac{\log T(r'_n, A_1)}{\log r'_n} = \alpha. \quad (5.5)$$

By $m(E) = \delta < +\infty$, there exists a point $r_n \in [r'_n, r'_n + \delta + 1] - E$. From

$$\frac{\log T(r_n, A_1)}{\log r_n} \geq \frac{\log T(r'_n, A_1)}{\log(r'_n + \delta + 1)} = \frac{\log T(r'_n, A_1)}{\log r'_n + \log(1 + (\delta + 1)/r'_n)} \quad (5.6)$$

we get

$$\liminf_{r_n \rightarrow \infty} \frac{\log T(r_n, A_1)}{\log r_n} \geq \alpha. \quad (5.7)$$

So, for any given ε with $0 < 2\varepsilon < \alpha - \beta$, and for $j = 0, 2, \dots, k-1$,

$$T(r_n, A_j) \leq r_n^{\beta+\varepsilon} \quad \text{and} \quad T(r_n, A_1) > r_n^{\alpha-\varepsilon} \quad (5.8)$$

hold for sufficiently large r_n . By (5.4) and (5.8), we get for sufficiently large r_n ,

$$r_n^{\alpha-\varepsilon} \leq (k-1)r_n^{\beta+\varepsilon} + 2T(r_n, f) + O(\log r_n). \quad (5.9)$$

Therefore,

$$\limsup_{r_n \rightarrow \infty} \frac{\log T(r_n, f)}{\log r_n} \geq \alpha - \varepsilon \quad (5.10)$$

and since ε is arbitrary, we get $\sigma(f) \geq \sigma(A_1) = \alpha$. This proves Theorem 1.7. \diamond

The next example illustrates Theorem 1.7.

Example 5.1 The equation

$$f'''' + e^{-z}f' - 16f = 0 \quad (5.11)$$

has a solution $f(z) = 120e^{2z} + 16e^z + 1$, where $\sigma(A_1) = 1$, $\sigma(A_j) = 0$, $j = 0, 2, 3$, and $\sigma(f) = 1$.

6 Proof of Theorem 1.8

Let $\beta < \sigma(A_0)$ and let $f \not\equiv 0$ be a solution of (1.4). Suppose that $\beta < \alpha < \sigma(A_0)$ and that there is a set $E_\beta \subset [0, +\infty)$ of lower density 1 satisfying (1.19). Set

$$E_1 = \{z : |z| = r \in E_\beta \text{ and } |A_j(z)| = \min_{|z|=r} |A_j(z)|, j = 1, 2, \dots, k-1\}. \quad (6.1)$$

Then $\underline{\text{dens}}\{|z| : z \in E_1\} = 1$ and

$$|A_j(z)| \leq \exp(r^\beta), \quad j = 1, 2, \dots, k-1 \quad (6.2)$$

for all $z \in E_1$. Also, from Lemma 2.4, there is a set $E_2 \subset [0, +\infty)$ of positive upper density such that for all z satisfying $|z| \in E_2$, we have

$$|A_0(z)| \geq \exp(r^\alpha). \quad (6.3)$$

Now let $E = \{z \in E_1 : |z| \in E_2\}$. Then, with a set E and the number α , $A_0(z), \dots, A_{k-1}(z)$ satisfy the hypothesis of Lemma 2.5 respectively. Hence we conclude by Lemma 2.5 that every solution $f \neq 0$ of (1.4) satisfies $\sigma(f) = +\infty$ and

$$\sigma_2(f) = \limsup_{r \rightarrow \infty} \frac{\log \log T(r, f)}{\log r} \geq \alpha. \quad (6.4)$$

Thus the result of the theorem follows since α is arbitrary. \diamond

The next example illustrates Theorem 1.8.

Example 6.1 Let $P_1(z), \dots, P_{k-1}(z)$ be nonconstant polynomials, and let $h_1(z), \dots, h_{k-1}(z)$ be entire functions satisfying $\sigma(h_j) < \deg P_j$ for $j = 1, \dots, k-1$. Let $A_0(z)$ be an entire function with $0 < \sigma(A_0) < 1/2$. Then, by Theorem 1.8, every solution $f \neq 0$ of the equation

$$f^{(k)} + h_{k-1}(z)e^{P_{k-1}(z)}f^{(k-1)} + \dots + h_1(z)e^{P_1(z)}f' + A_0(z)f = 0 \quad (6.5)$$

is of infinite order with $\sigma_2(f) \geq \sigma(A_0)$, because

$$\min_{|z|=r} |h_j(z)e^{P_j(z)}| \rightarrow 0, \quad j = 1, \dots, k-1$$

as $r \rightarrow \infty$.

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References

- [1] B. Belaïdi, *Estimation of the hyper-order of entire solutions of complex linear ordinary differential equations whose coefficients are entire functions*, E. J. Qualitative Theory of Diff. Equ., No. 5, (2002), 8 pp.
- [2] A. Besicovitch, *On integral functions of order < 1* , Math. Ann., **97** (1927), pp. 677-695.
- [3] G. Gundersen, *Finite order solutions of second order linear differential equations*, Trans. Amer. Math. Soc. 305 (1988), pp. 415-429.
- [4] G. Gundersen, *Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates*, J. London Math. Soc. (2) 37 (1988), pp. 88-104.

- [5] G. Gundersen, M. Steinbart, and S. Wang, *The possible orders of solutions of linear differential equations with polynomial coefficients*, Trans. Amer. Math. Soc. **350** (1998), pp. 1225-1247.
- [6] W. K. Hayman, *Meromorphic functions*, Clarendon Press, Oxford, 1964.
- [7] K. H. Kwon, *On the growth of entire functions satisfying second order linear differential equations*, Bull. Korean Math. Soc. **33** (1996), No. 3, pp. 487-496.
- [8] A. I. Markushevich, *Theory of functions of a complex variable*, Vol. II, translated by R. A. Silverman, Prentice-Hall, Englewood Cliffs, New Jersey, 1965.
- [9] G. Valiron, *Lectures on the general theory of integral functions*, translated by E. F. Collingwood, Chelsea, New York, 1949.
- [10] H. Wittich, *Neuere Untersuchungen über eindeutige analytische Funktionen*, 2nd Edition, Springer-Verlag, Berlin-Heidelberg-New York, 1968.

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