On certain nonlinear elliptic systems with indefinite terms *

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Abstract

We consider an elliptic quasi linear system with indefinite term on a bounded domain. Under suitable conditions, existence and positivity results for solutions are given.

1 Introduction

The purpose of this article is to find positive solutions to the system

\[-\Delta_p u = m(x) \frac{\partial H}{\partial u}(u,v) \quad \text{in } \Omega\]
\[-\Delta_q v = m(x) \frac{\partial H}{\partial v}(u,v) \quad \text{in } \Omega\]
\[u = v = 0 \quad \text{on } \partial \Omega\]

where \(\Omega\) is a bounded regular domain of \(\mathbb{R}^N\), with a smooth boundary \(\partial \Omega\), \(\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)\) is the \(p\)-Laplacian with \(1 < p < N\), \(m\) is a continuous function on \(\Omega\) which changes sign, and \(H\) is a potential function which will be specified later.

The case where the sign of \(m\) does not change has been studied by F. de Thélin and J. Vélin [9]. These authors treat the system (1.1) with a function \(H\) having the following properties

- There exists \(C > 0\), for all \(x \in \Omega\), for all \((u,v) \in D_3\) such that \(0 \leq H(x,u,v) \leq C(|u|^{p'} + |v|^{q'})\)
- There exists \(C' > 0\), for all \(x \in \Omega\), for all \((u,v) \in D_2\) such that \(H(x,u,v) \leq C'\)
- There exists a positive function \(a\) in \(L^\infty(\Omega)\), such that for each \(x \in \Omega\) and \((u,v) \in D_1 \cap \mathbb{R}^2_+\), \(H(x,u,v) = a(x)u^{\alpha+1}v^{\beta+1}\),

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where
\[ D_1 = \{(u, v) \in \mathbb{R}^2 : |u| \geq A \text{ or } |v| \geq A\}, \]
\[ D_2 = \{(u, v) \in \mathbb{R}^2 \setminus D_1 : |u| \geq \delta \text{ or } |v| \geq \delta\}, \]
and \( D_3 = \mathbb{R}^2 \setminus (D_1 \cup D_2) \) with \( A > \delta > 0 \), \( 1 < p' < p^* := \frac{Np}{N-p} \), and \( 1 < q' < q^* \).

They established the existence results under the conditions
\[ \alpha + 1 > \frac{1}{p} + \frac{1}{q} > 1 \quad \text{and} \quad \alpha + 1 > \frac{1}{p^*} + \frac{1}{q^*} > 1 \]
\[ H(u, v) = a|u|^\gamma + c|v|^\delta + b|u|^\alpha|v|^\beta \]
where \( \alpha, \beta \geq 0; \gamma, \delta > 1 \) and \( a, b \) and \( c \) are real numbers. The case where the system (1.1) is governed by a single operator \( \Delta_p \) has been studied by Baghli [3].

Our aim is to extend to the system (1.1) the results obtained in the scalar case (see [5]). Our existence results follow from modified quasilinear system in order to apply the Palais-Smale condition (P.S.) and then the Moser’s Iterative Scheme as in T. Ôtani [6] or in F. de Thélín and J. Vélin [9]. We consider only weak solutions, and assume that \( H \) satisfies the following hypothesis.

(H1) \( H \in C^1(\mathbb{R}^+ \times \mathbb{R}^+) \)
(H2) \( H(u, v) = o(u^p + v^q) \) as \( (u, v) \to (0^+, 0^+) \)
(H3) There exists \( R_0 > 0 \) and \( \mu, 1 < \mu < \min(p^*/p, q^*/q) \), such that
\[ \frac{u}{p} \frac{\partial H}{\partial u}(u, v) + \frac{v}{q} \frac{\partial H}{\partial v}(u, v) \geq \mu H(u, v) > 0 \quad \forall (u, v) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+, \quad u^p + v^q \geq R_0. \]

2 Preliminaries and existence results

The values of \( H(u, v) \) are irrelevant for \( u \leq 0 \) or \( v \leq 0 \). We set
\[ I(u, v) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx + \frac{1}{q} \int_{\Omega} |\nabla v|^q dx + \int_{\Omega} m(x) H(u, v) dx \]
defined on \( E := W^1_0(u, v) \times W^1_0(v) \). The solutions of the system (1.1) are critical points of the functional \( I \). Note that the functional \( I \) does not satisfy in general the Palais-Smale condition since
\[ B_{\mu}H(u, v) := \frac{u}{p} \frac{\partial H}{\partial u}(u, v) + \frac{v}{q} \frac{\partial H}{\partial v}(u, v) - \mu H(u, v) \]
is not always bounded. In order to apply Ambrosetti-Rabinowitz Theorem [2], we modify $H$ so that the corresponding $B_\mu H(u, v)$ becomes bounded. Let

$$A(R) = \max \left\{ \frac{H(u, v)}{(u^p + v^q)^\mu} : R \leq u^p + v^q \leq R + 1 \right\}$$

and

$$C_R = \max \left\{ \sup_{u^p + v^q \leq R + 1} \left| \frac{\partial H}{\partial u}(u, v) \right| + 2p\mu A(R)(R + 1)^{\mu+1-\frac{1}{4}} \sup_{R \leq r \leq R + 1} |\eta'_R(r)|; \right.$$  

$$\sup_{u^p + v^q \leq R + 1} \left| \frac{\partial H}{\partial v}(u, v) \right| + 2q\mu A(R)(R + 1)^{\mu+1-\frac{1}{4}} \sup_{R \leq r \leq R + 1} |\eta'_R(r)| \right\}$$

where $\eta_R \in C^1(\mathbb{R})$ is a cutting function defined by

$$\eta_R(r) = \begin{cases} 
1 & \text{if } r \leq R \\
< 0 & \text{if } R < r < R + 1 \\
0 & \text{if } r \geq R + 1.
\end{cases}$$

Our main result is the following:

**Theorem 2.1** Assume that $(H_i)_{i=1,2,3}$ hold and $C_R = o(R^{p^*p'q^{-\frac{q}{q'-q}}\mu})$ for $R$ sufficiently large. Then the system (1.1) has at least one nontrivial solution $(u, v)$ in $E \cap [L^\infty(\Omega)]^2$ with $u$ and $v$ positive.

Before proving this theorem, we truncate the potential function $H$.

**The modified problem**

Let $R \geq R_0$ be fixed, and set

$$H_R(u, v) := \eta_R(u^p + v^q)H(u, v) + (1 - \eta_R(u^p + v^q))A(R)(u^p + v^q)^\mu,$$

By construction $H_R$ is $C^1$ and nonnegative. Let

$$M_R := (R + 1)^{\mu} \max_{u^p + v^q \leq R + 1} \left[ \eta'_R(u^p + v^q)(H(u, v) - A(R)(u^p + v^q)^\mu) \right]$$

$$+ \max_{u^p + v^q \leq R + 1} B_\mu H(u, v),$$

**Lemma 2.2** $H_R$ satisfies (H1)-(H3) and the following estimates

$$0 \leq B_\mu H_R(u, v) \leq M_R, \quad \forall (u, v) \in \mathbb{R}^+ \times \mathbb{R}^+,$$  

$$\left| \frac{\partial H_R}{\partial u}(u, v) \right| \leq C_R + \mu p A(R)u^{p-1}(u^p + v^q)^{\mu-1},$$  

$$\left| \frac{\partial H_R}{\partial v}(u, v) \right| \leq C_R + \mu q A(R)v^{q-1}(u^p + v^q)^{\mu-1}, \quad \forall (u, v) \in \mathbb{R}_+^2,$$

$$H_R(u, v) \geq \frac{m_{R_0}}{R_0} (u^p + v^q)^\mu \quad \forall (u, v) \in \mathbb{R}_+ \times \mathbb{R}_+^*, \text{ such that } u^p + v^q \geq R_0,$$

with $m_{R_0} := \min\{H(u, v); u^p + v^q = R_0\}$. 

Proof. (H1) and (H2) can be easily verified for $H_R$. We verify for (H3) as follows: For any $\nu > 1$, we have

$$B_\nu H_R(u, v) = (u^p + v^q)\eta'_R(u^p + v^q)[H(u, v) - A(R)(u^p + v^q)] + \eta_R(u^p + v^q) B_\nu H(u, v),$$

for $R_0 \leq u^p + v^q \leq R$;

$$B_\nu H_R(u, v) = B_\nu H(u, v) \geq B_\mu H(u, v) \geq 0 \quad \text{for } 1 < \nu \leq \mu$$

for $R \leq u^p + v^q \leq R + 1$;

$$B_\nu H_R(u, v) \geq \eta_R(u^p + v^q) B_\nu H(u, v) \geq \eta_R(u^p + v^q) B_\mu H(u, v) \geq 0 \quad \text{for } 1 < \nu \leq \mu;$$

finally for $u^p + v^q \geq R + 1$, $B_\nu H_R(u, v) = 0$ for any $\nu > 1$. Thus (H3) holds for $H_R$.

Conditions (2.1) and (2.2) result from straightforward computations. Using (H3), we have

$$H_R(u, v) \geq \frac{m R_0}{R_0^p} (u^p + v^q)^\mu, \forall (u, v) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \text{ such that } u^p + v^q \geq R_0. \quad (2.4)$$

In fact, put $f(t) := H_R(t^{1/p}u, t^{1/q}v)$ with $u^p + v^q \geq R_0$ then

$$f'(t) = \frac{1}{t} \left[ t^{1/p} \frac{\partial H_R}{\partial u}(t^{1/p}u, t^{1/q}v) + \frac{t^{1/q}v}{q} \frac{\partial H_R}{\partial v}(t^{1/p}u, t^{1/q}v) \right] \geq \frac{\mu}{t} f(t) \quad \text{for all } t \geq t_0 := \frac{R_0}{u^p + v^q}(\leq 1). \quad (2.5)$$

Integrating (2.5) between $t_0$ and $t$, we obtain

$$\frac{f(t)}{f(t_0)} \geq \frac{t^\mu}{t_0^\mu} \quad \text{for all } t \geq t_0 \quad (2.6)$$

and taking $t = 1$ in (2.6), we have

$$H_R(u, v) = f(1) \geq \frac{(u^p + v^q)^\mu}{R_0^\mu} f(t_0)$$

and $f(t_0) = H_R(u_1, v_1) = H(u_1, v_1)$, where $u_1 = (\frac{R_0}{u^p + v^q})^{1/p}u$, and

$$v_1 = (\frac{R_0}{u^p + v^q})^{1/q}v.$$ Consequently,

$$\min_{u^p + v^q \geq R_0} f(t_0(u, v)) = \min_{u^p + v^q = R_0} H(u, v),$$

hence (2.4) follows. Now, consider the modified system

$$-\Delta p u = m(x) \frac{\partial H_R}{\partial u}(u, v) \quad \text{in } \Omega$$

$$-\Delta q v = m(x) \frac{\partial H_R}{\partial v}(u, v) \quad \text{in } \Omega$$

$$u = v = 0 \quad \text{on } \partial \Omega \quad (2.7)$$
which has an associated functional \( I_R \) defined on \( E \) as

\[
I_R(u, v) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx + \frac{1}{q} \int_{\Omega} |\nabla v|^q dx - \int_{\Omega} m(x) R(u, v) dx.
\]

**Lemma 2.3** Under the hypotheses (H1)-(H3), the functional \( I_R \) satisfies the Palais-Smale condition.

**Proof.** Let \((u_n, v_n)\) be an element of \( E \) such that \( I_R(u_n, v_n) \) is bounded and \( I'_R(u_n, v_n) \rightarrow 0 \) strongly in \( W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega) \) (dual space of \( E \)).

Claim 1. \((u_n, v_n)\) is bounded in \( E \). In fact, for any \( M \), we have

\[
-M \leq \frac{1}{p} \int_{\Omega} |\nabla u_n|^p dx + \frac{1}{q} \int_{\Omega} |\nabla v_n|^q dx - \int_{\Omega} m(x) R(u_n, v_n) dx \leq M;
\]

and for \( \varepsilon \in (0, 1) \), we have again

\[
-M - \varepsilon \leq \frac{1}{p} \int_{\Omega} |\nabla u_n|^p dx + \frac{1}{q} \int_{\Omega} |\nabla v_n|^q dx - \int_{\Omega} m(x) [u_n \frac{\partial R}{\partial u}(u_n, v_n) + v_n \frac{\partial R}{\partial v}(u_n, v_n)] dx \leq \varepsilon.
\]

Then we obtain

\[
\frac{\mu - 1}{p} \int_{\Omega} |\nabla u_n|^p dx + \frac{\mu - 1}{q} \int_{\Omega} |\nabla v_n|^q dx \leq M \mu - \int_{\Omega} m(x) B \| R(u, v) \| dx \\
\leq M \mu + 1 + |m|_0 M(\text{meas } \Omega)
\]

where \( |m|_0 := \max_{x \in \Omega} (|m(x)|) \). Hence \((u_n, v_n)\) is bounded in \( E \).

Claim 2. \((u_n, v_n)\) converges strongly in \( E \). Since \((u_n, v_n)\) is bounded in \( E \), there exists a subsequence denoted again by \((u_n, v_n)\) which converges weakly in \( E \) and strongly in the space \( L^q(\Omega) \times L^r(\Omega) \) for any \( \zeta \) and \( \eta \) such that, \( 1 < \zeta < p^* \) and \( 1 < \eta < q^* \). From the definition of \( I'_R \), we write

\[
\int_{\Omega} (|\nabla u_n|^p \nabla u_n - |\nabla u_t|^p \nabla u_t) \nabla (u_n - u_t) dx \\
= \langle I'_R(u_n, v_n) - I'_R(u_t, v_t), (u_n - u_t, 0) \rangle \\
+ \int_{\Omega} m(x) \left[ \frac{\partial R}{\partial u}(u_n, v_n) - \frac{\partial R}{\partial u}(u_t, v_t) \right] (u_n - u_t) dx.
\]

By assumptions on \( I'_R \), \( \langle I'_R(u_n, v_n) - I'_R(u_t, v_t), (u_n - u_t, 0) \rangle \) converges to 0 as \( n \) and \( t \) tend to \( +\infty \). In what follows, \( C \) denotes a generic positive constant. Now, we prove that

\[
C_{n,t} := \int_{\Omega} m(x) \left[ \frac{\partial R}{\partial u}(u_n, v_n) - \frac{\partial R}{\partial u}(u_t, v_t) \right] (u_n - u_t) dx
\]
Using Hölder’s inequality and Sobolev’s embeddings, we obtain

$$|C_{n,l}| \leq |m|_0 \int_\Omega \left| \frac{\partial H_R}{\partial u}(u_n, v_n) \right| + \left| \frac{\partial H_R}{\partial u}(u_l, v_l) \right| |u_n - u_l| \, dx$$

and

$$\int_\Omega \left| \frac{\partial H_R}{\partial u}(u_n, v_n) \right| |u_n - u_l| \, dx$$

$$\leq \int_\Omega (C_R + \mu pA(R))|u_n|^{p-1}(|u_n|^p + |v_n|^q)^{q-1})|u_n - u_l| \, dx$$

$$\leq 2^{\mu-1}C_R \int_\Omega (1 + |u_n|^{p-1} + |u_n|^{p-1}|v_n|^{q-1})|u_n - u_l| \, dx$$

$$\leq 2^{\mu-1}C_R \left[ \int_\Omega |u_n - u_l| \, dx + \int_\Omega |u_n|^{p-1} |u_n - u_l| \, dx \right.$$  

$$+ \int_\Omega |u_n|^{p-1} |v_n|^{q-1} |u_n - u_l| \, dx \right] .$$

Using Hölder’s inequality and Sobolev’s embeddings, we obtain

$$\int_\Omega \left| \frac{\partial H_R}{\partial u}(u_n, v_n) \right| |u_n - u_l| \, dx$$

$$\leq 2^{\mu-1}C_R (\text{meas } \Omega)^{\frac{\mu-1}{\mu}} \left[ \int_\Omega |u_n - u_l|^p \, dx \right]^{1/p}$$

$$+ 2^{\mu-1}C_R \left[ \int_\Omega |u_n|^p \, dx \right]^{\frac{\mu-1}{\mu}} \left[ \int_\Omega |u_n - u_l|^p \, dx \right]^{\frac{1}{p}}$$

$$+ 2^{\mu-1}C_R \left[ \int_\Omega |v_n|^q \, dx \right]^{\frac{\mu-1}{\mu}} \left[ \int_\Omega |u_n - u_l|^p \, dx \right]^{\frac{1}{p}},$$

(because $(u_n) \in W^{1,p}_0(\Omega)$ and $\mu p < p^*$, $(v_n) \in W^{1,q}_0(\Omega)$ and $\mu q < q^*$). Then

$$\int_\Omega \left| \frac{\partial H_R}{\partial u}(u_n, v_n) \right| |u_n - u_l| \, dx \leq C\|u_n - u_l\|_{L^p(\Omega)} + C\|u_n - u_l\|_{L^p(\Omega)}.$$

Similarly, we obtain

$$\int_\Omega \left| \frac{\partial H_R}{\partial u}(u_l, v_l) \right| |u_n - u_l| \, dx \leq C\|u_n - u_l\|_{L^p(\Omega)} + C\|u_n - u_l\|_{L^p(\Omega)},$$

and so $|C_{n,l}| \leq |m|_0(C\|u_n - u_l\|_{L^p(\Omega)} + C\|u_n - u_l\|_{L^p(\Omega)})$. Hence $C_{n,l}$ converges to 0 as $n$ and $l$ tend to $+\infty$.

We have the following algebraic relation [8]

$$|\nabla u_n - \nabla u_l|^p$$

$$\leq C \left[ (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_l|^{p-2} \nabla u_l) \nabla (u_n - u_l) \right]^{\frac{p}{2}} \left( |\nabla u_n|^p + |\nabla u_l|^p \right)^{1 - \frac{2}{p}} ,$$

(2.8)
where \( s = \begin{cases} p & \text{for } 1 < p \leq 2 \\ 2 & \text{for } 2 < p \end{cases} \). Integrating (2.8) on \( \Omega \), and using Hölder’s inequality in the right hand side, we obtain

\[
\|u_n - u_l\|_{1,p} \leq C \left[ \int_{\Omega} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_l|^{p-2}\nabla u_l)\nabla (u_n - u_l)dx \right]^\frac{s}{2} \left( \|u_n\|_{1,p}^p + \|u_l\|_{1,p}^p \right)^{1-s}.
\]

Now since

\[
\int_{\Omega} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_l|^{p-2}\nabla u_l)\nabla (u_n - u_l)dx \to 0
\]
as \( n \) and \( l \) tend to \( +\infty \), the sequence \((u_n)\) converges strongly in \( W^{1,p}_0(\Omega) \). Similarly we prove that the sequence \((v_n)\) converges strongly in \( W^{1,q}_0(\Omega) \).

The next lemma shows that \( I_R \) satisfies the geometric assumptions of the Mountain-Pass Theorem.

**Proposition 2.4** Under assumptions (H1)-(H3) we have

1. There exist two positive real numbers \( \rho, \sigma \) and a neighborhood \( V_\rho \) of the origin of \( E \), such that for any element \((u,v)\) on the boundary of \( V_\rho \):

\[
I_R(u,v) \geq \sigma > 0.
\]

2. There exist \((\phi, \theta)\) in \( E \) such that

\[
I_R(\phi, \theta) < 0.
\]

**Proof.** From (H2) and taking into account that \( H_R(u,v) = H(u,v) \) for \( u^p + v^q \leq R \), we can write

\[
\forall \varepsilon > 0, \exists \delta_\varepsilon > 0 : u^p + v^q \leq \delta_\varepsilon \implies H_R(u,v) \leq \varepsilon (u^p + v^q),
\]

and since \( H_R(u,v)/(u^p + v^q)^\mu \) is uniformly bounded as \( u^p + v^q \) tends to \( +\infty \)

\[
\exists M(\varepsilon, R) > 0 : u^p + v^q \geq \delta_\varepsilon \implies H_R(u,v) \leq M(u^p + v^q)^\mu.
\]

Then for every \((u,v)\) in \( \mathbb{R}^+ \times \mathbb{R}^+ \) we have

\[
H_R(u,v) \leq \varepsilon (u^p + v^q) + M(u^p + v^q)^\mu.
\]

Hence

\[
\int_{\Omega} m(x)H_R(u,v)dx \\
\leq \ |m|_0 \left[ \varepsilon \int_{\Omega} (u^p + v^q)dx + M \int_{\Omega} (u^p + v^q)^\mu dx \right] \\
\leq \ |m|_0 \left[ \int_{\Omega} (\varepsilon u^p + 2^{\mu-1}Mu^p)dx + \int_{\Omega} (\varepsilon v^q + 2^{\mu-1}Mv^q)dx \right] \\
\leq \ C|m|_0 \left[ \varepsilon (\|u\|^p_{1,p} + \|v\|^q_{1,q}) + M(\|u\|^{\mu p}_{1,p} + \|v\|^{\mu q}_{1,q}) \right]
\]
Proposition 3.1

Under the assumptions of Theorem 2.1, there exist two sequences \((\phi, \theta)\) with 
\[
\lim_{t \to \infty} I_R(\phi, \theta) = -\infty,
\]
for every \((u, v)\) in the sphere \(S(0, \rho)\) of \(E\) where \(\rho\) is such that \(0 < \rho < \min(\rho_1, \rho_2)\) with
\[
\rho_1 = \left[\frac{1}{pMC|m_0|} - \frac{\varepsilon}{M}\right]^\frac{1}{p-\sigma} \quad \text{and} \quad \rho_2 = \left[\frac{1}{qMC|m_0|} - \frac{\varepsilon}{M}\right]^\frac{1}{q-\sigma}
\]
with \(\varepsilon\) sufficiently small.

2. Choose \((\phi, \theta) \in E\) such that: \(\phi > 0, \theta > 0,\)
\[
\text{supp } \phi \subset \Omega^+, \quad \text{supp } \theta \subset \Omega^+,
\]
where \(\Omega^+ = \{x \in \Omega; m(x) > 0\}\). Hence, for \(t\) sufficiently large,
\[
I_R(t^{1/p}\phi, t^{1/q}\theta) = \frac{t}{p} \|\phi\|_{1,p}^p + \frac{t}{q} \|\theta\|_{1,q}^q - \int_\Omega m(x) H_R(t^{1/p}\phi, t^{1/q}\theta) dx
\]
\[
\leq t \left[\|\phi\|_{1,p}^p + \|\theta\|_{1,q}^q \right] - \mu \frac{m_{R_0}}{R_0^\mu} \int_\Omega m(x)(\phi^{\mu} + \theta^{\mu}) dx
\]
and so \(\lim_{t \to +\infty} I_R(t^{1/p}\phi, t^{1/q}\theta) = -\infty\), (because \(\mu > 1\)). By continuity of \(I_R\) on \(E\), there exists \((\phi, \theta)\) in \(E \setminus B(0, \rho)\) such that \(I_R(\phi, \theta) < 0\). By the usual Mountain-Pass Theorem, we know that there exists a critical point of \(I_R\) which we denote by \((u_R, v_R)\), and corresponding to a critical value \(c_R \geq \sigma\). Since \((u^+_R, v^+_R)\), where \(u^+_R = \max(u_R, 0)\), is also solution for the system \((S^R)\), we assume \(u_R \geq 0\) and \(v_R \geq 0\). Positivity of \(u_R\) and \(v_R\) follows from Harnack’s inequality (see J. Serrin [7]). We prove now that, under some conditions, \((u_R, v_R)\) is also solution of the system (2.7).

3 Existence results

We adapt the Moser iteration used in [6, 9] to construct two strictly unbounded sequences \((\lambda_k)_{k \in \mathbb{N}}\) and \((\mu_k)_{k \in \mathbb{N}}\) such that \((u_R, v_R)\) satisfies
\[
\begin{cases}
u \in L^{\lambda_k}(\Omega) \\
v \in L^{\mu_k}(\Omega)
\end{cases}
\]
then
\[
\begin{cases}
u \in L^{\lambda_{k+1}}(\Omega) \\
v \in L^{\mu_{k+1}}(\Omega)
\end{cases}
\]

Bootstrap argument

**Proposition 3.1** Under the assumptions of Theorem 2.1, there exist two sequences \((\lambda_k)_{k \in \mathbb{N}}\) and \((\mu_k)_{k \in \mathbb{N}}\) such that

1. For each \(k\), \(u_R\) and \(v_R\) belong to \(L^{\lambda_k}(\Omega)\) and \(L^{\mu_k}(\Omega)\) respectively
2. There exist two positive constants $C_p$ and $C_q$ such that

$$\|u_R\|_\infty \leq \limsup_{k \to +\infty} \|u_R\|_{L^{\lambda_k}} \leq C_p, \quad \text{and} \quad \|v_R\|_\infty \leq \limsup_{k \to +\infty} \|v_R\|_{L^{\mu_k}} \leq C_q.$$

Lemma 3.2 Let $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ be two positive sequences satisfying, for each integer $k$, the relations

$$\frac{p + a_k}{\lambda_k} + \frac{q(\mu - 1)}{\mu_k} = 1, \quad \text{and} \quad \frac{q + b_k}{\lambda_k} + \frac{p(\mu - 1)}{\lambda_k} = 1. \quad (3.1)$$

If $u_R$ and $v_R$ are in $L^{\lambda_k}(\Omega)$ and $L^{\mu_k}(\Omega)$ respectively, $\lambda_{k+1} \leq (1 + \frac{a_k}{p})\pi_p$, $\mu_{k+1} \leq (1 + \frac{b_k}{q})\pi_q$ with $1 < \pi_p < p^*$ and $1 < \pi_q < q^*$, then we have:

$$\|u_R\|_{\lambda_{k+1}} \leq K_p \left\{ \theta_p \left[ 1 + \frac{a_k}{p} \right] \left[ C_R |m|_0 (\|u_R\|_{\lambda_k} + \|v_R\|_{\mu_k}) \right]^{1/p} \right\}^{\lambda_{k+1}}, \quad (3.2)$$

$$\|v_R\|_{\mu_{k+1}} \leq K_q \left\{ \theta_q \left[ 1 + \frac{b_k}{q} \right] \left[ C_R |m|_0 (\|u_R\|_{\lambda_k} + \|v_R\|_{\mu_k}) \right]^{1/q} \right\}^{\mu_{k+1}} \quad (3.3)$$

where $\|z\|_{\beta}$ is $\|z\|_{L^\beta(\Omega)}$ and $K_p$, $K_q$, $\theta_p$, and $\theta_q$ are positive constants.

Proof. Remark that if, for an infinite number of integers $k$, $\|u_R\|_{\lambda_k} \leq 1$ then $\|u_R\|_{\infty} \leq 1$ and proposition 1 is proved. So we suppose that $\|u_R\|_{\lambda_k} \geq 1$ for all $k \in \mathbb{N}$. Let $\zeta_n, n \in \mathbb{N}$, be $C^1$ functions such that

$$\zeta_n(s) = s \quad \text{if} \quad s \leq n$$

$$\zeta_n(s) = n + 1 \quad \text{if} \quad s \geq n + 2$$

$$0 < \zeta_n'(s) < 1 \quad \text{if} \quad s \in \mathbb{R}^+.$$ 

Put $u_n := \zeta_n(u_R)$, then $u_n^{1+a_k} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $u_R$ satisfies the first equation of the system (2.7). Multiply this equation by $u_n^{1+a_k}$ and integrate over $\Omega$ to get

$$\int_\Omega -\Delta_p u_{R} u_n^{1+a_k} dx = \int_\Omega m(x) \frac{\partial H_R}{\partial a} (u_R, v_R) u_n^{1+a_k} dx$$

$$\leq 2^\mu - 1 C_R |m|_0 \int_\Omega (1 + u_R^{\mu - 1} + u_R^{\mu - 1} v_R^{\mu - q}) u_n^{1+a_k} dx.$$

Since $u_n \leq u_R$, we have

$$\int_\Omega -\Delta_p u_{R} u_n^{1+a_k} dx$$

$$\leq 2^{\mu - 1} C_R |m|_0 \left\{ \int_\Omega u_n^{1+a_k} dx + \int_\Omega u_R^{\mu + a_k} dx + \int_\Omega u_R^{\mu + a_k} v_R^{\mu - q} dx \right\}.$$ 

Using Hölder’s inequality, we obtain

$$\int_\Omega -\Delta_p u_{R} u_n^{1+a_k} dx \leq 2^{\mu - 1} C_R |m|_0 \left\{ (\text{meas} \Omega)^\frac{1+a_k}{\lambda_k} \|u_R\|_{\lambda_k}^{1+a_k} + \|u_R\|_{\mu_k}^{\mu + a_k} + \|u_R\|_{\lambda_k}^{\mu + a_k} \right\}.$$
We shall show below that $p\mu + a_k = \lambda_k$. Since $\|u_R\|_{\lambda_k} \geq 1$, we get
\[
\int_{\Omega} -\Delta_p u_R u_n^{1+a_k} \, dx \leq 2^{n-1} C_R |m|_0 \max(1, \text{meas } \Omega) \left[ 2 \| u_R \|_{\lambda_k}^{p+a_k} + \| v_R \|_{\mu_k}^{q-a-q} \right].
\]
Moreover, using the relation (3.1), we obtain
\[
\| u_R \|_{\lambda_k}^{p+a_k} \| v_R \|_{\mu_k}^{q-a-q} \leq \| u_R \|_{\lambda_k}^{\lambda_k} + \| v_R \|_{\mu_k}^{\mu_k},
\]
so, with $c_0 := 3 \max(1, \text{meas } \Omega)$,
\[
\int_{\Omega} -\Delta_p u_R u_n^{1+a_k} \, dx \leq 2^{n-1} c_0 C_R |m|_0 \left[ \| u_R \|_{\lambda_k}^{\lambda_k} + \| v_R \|_{\mu_k}^{\mu_k} \right].
\] (3.4)

On the other hand we have
\[
\int_{\Omega} -\Delta_p u_R u_n^{1+a_k} \, dx = (1 + a_k) \int_{\Omega} |\nabla u_R|^p \zeta_n'(u_R) u_n^{a_k} \, dx \geq (1 + a_k) \int_{\Omega} |\nabla u_R|^p (\zeta_n'(u_R)) u_n^{a_k} \, dx = (1 + a_k) \int_{\Omega} |\nabla u_n|^p u_n^{a_k} \, dx
\]
and thus
\[
\int_{\Omega} -\Delta_p u_R u_n^{1+a_k} \, dx \geq \int_{\Omega} |\nabla u_n|^p u_n^{a_k} \, dx.
\] (3.5)

Since $u_n^{1+\frac{a_k}{p}} \in W_0^{1,p}(\Omega)$, the continuous imbedding of $W_0^{1,p}(\Omega)$ in $L^{\pi_p}(\Omega)$ implies the existence of a positive constant $c$ such that
\[
\left( \int_{\Omega} |u_n^{1+\frac{a_k}{p}}|^{\pi_p} \, dx \right)^{\frac{1}{\pi_p}} \leq c \left( \int_{\Omega} |\nabla u_n^{1+\frac{a_k}{p}}|^p \, dx \right)^{\frac{1}{p}} = c \left[ 1 + \frac{a_k}{p} \right] \left( \int_{\Omega} |\nabla u_n|^{\pi_p} \, dx \right)^{\frac{1}{p}}. \tag{3.6}
\]

By assumption, we have $\lambda_k+1 \leq j_k := [1 + \frac{a_k}{p}] \pi_p$. Then
\[
\| u_n \|_{\lambda_k+1} \leq (\text{meas } \Omega)^{m_k} \| u_n \|_{j_k}, \quad \text{where } m_k := \frac{1}{\lambda_k+1} - \frac{1}{(1 + \frac{a_k}{p}) \pi_p}
\]
and thus
\[
\| u_n \|_{\lambda_k+1} \leq K_p \| u_n \|_{j_k}^{\lambda_k+1}
\]
where $K_p$ is a positive constant greater than $(\text{meas } \Omega)^{m_k \lambda_k+1}$ independently of the integer $k$. By the relation (3.6),
\[
\| u_n \|_{j_k}^{\lambda_k+1} \leq \left[ c \left[ 1 + \frac{a_k}{p} \right] \left( \int_{\Omega} |\nabla u_n|^{\pi_p} \, dx \right)^{\frac{1}{p}} \right]^{\frac{\lambda_k+1}{\pi_p}} \tag{3.7}
\]
Combining the inequalities (3.4)-(3.7), we deduce
\[
\|u_n\|_{\lambda_{k+1}^{\lambda_{k+1}}} \leq K_p \left\{ \theta_p \left[ 1 + \frac{a_k}{p} \right] \{ C_R |m|_0 (\|u_R\|_{\lambda_k^{\lambda_k}} + \|v_R\|_{\mu_k^{\mu_k}}) \}^{1/p} \right\}^{\frac{\lambda_{k+1}}{1 + \frac{a_k}{p}}},
\]
with \( \theta_p = 2^{\frac{1}{p}} - 1 \). Hence, by letting \( n \to +\infty \), we obtain (3.2). Similarly we show (3.3).

**Construction and definition of \((\lambda_k)_k\) and \((\mu_k)_k\).**

Here we construct the sequences \((\lambda_k)_k\) and \((\mu_k)_k\) using tools similar as those in [O] or [TV]. The first terms of each sequence cannot be determined directly by using the Rellich-Kondrachov continuous imbedding result. So, we first construct two other sequences \((\hat{\lambda}_k)_k\) and \((\hat{\mu}_k)_k\), such that for each \(k\), \(u_R\) and \(v_R\) belong to \(L^{\hat{\lambda}_k}(\Omega)\) and \(L^{\hat{\mu}_k}(\Omega)\) respectively. By a suitable choice of \(k_0\), \(\hat{\lambda}_{k_0}\) and \(\hat{\mu}_{k_0}\) determine the first terms of \((\lambda_k)_k\) and \((\mu_k)_k\).

**Construction of \((\hat{\lambda}_k)_k\) and \((\hat{\mu}_k)_k\).**

Suppose \(p \leq q\), and fix a number \(s\), such that \(cp/p^* < s < 1/\mu\). Put
\[
\hat{C} := \frac{1}{2} + \frac{s}{2} \frac{p^*}{p}.
\]
Remark that \(\hat{C} > 1, 1 < \mu p \hat{C} < p^*\) and \(1 < \mu q \hat{C} < q^*\). Now, we take \(\hat{\lambda}_k = \mu p \hat{C}^k\) and \(\hat{\mu}_k = \mu q \hat{C}^k\). By definition of \((a_k)_k\), we have
\[
\frac{p + a_k}{\lambda_k} + \frac{\mu - 1}{\mu_k} q = 1
\]
then \(a_k = \hat{\lambda}_k - \mu p\). Similarly, we find \(b_k = \hat{\mu}_k - \mu q\).

**Lemma 3.3** For each integer \(k\), \(u_R \in L^{\hat{\lambda}_k}(\Omega)\) and \(v_R \in L^{\hat{\mu}_k}(\Omega)\).

**Proof.** By induction. For \(k = 0\), \(\hat{\lambda}_0 = \mu p < p^*\), \(\hat{\mu}_0 = \mu q < q^*\), and since \((u_R,v_R) \in E\), by the Sobolev imbedding theorem, we have \(u_R \in L^{\lambda_0}(\Omega)\) and \(v_R \in L^{\mu_0}(\Omega)\).

Suppose that the proposition is true for all integers \(k'\) such that \(0 \leq k' \leq k\). Take
\[
\pi_p = \mu p \hat{C} \quad \text{and} \quad \pi_q = \mu q \hat{C}.
\]
Since \(u_R \in L^{\hat{\lambda}_k}(\Omega)\) and
\[
[1 + \frac{a_k}{p}] \pi_p = [1 + \frac{\hat{\lambda}_k - \mu p}{p}] \mu p \hat{C} = \mu p \hat{C} + \mu^2 p \hat{C}^{k+1} - \mu^2 p \hat{C} \geq \mu p \hat{C}^{k+1}
\]
i.e. \(1 + \frac{2a_k}{p} \pi_p \geq \hat{\lambda}_{k+1}\), Lemma 3 allows us to write \(u_R \in L^{\hat{\lambda}_{k+1}}(\Omega)\) and \(v_R \in L^{\hat{\mu}_{k+1}}(\Omega)\).
Construction of \((\lambda_k)_k\) and \((\mu_k)_k\) \hspace{1cm} \text{Put}

\[
C = \frac{N}{N-p}, \quad \text{and} \quad \delta = \left[\frac{p}{N} \mu \tilde{C}^{k_0} - (\mu - 1)\right] C,
\]

where the integer \(k_0\) is chosen so as to have \(\delta > 0\). The sequences \((\lambda_k)_k\) and \((\mu_k)_k\) are defined by \(\lambda_k = pf_k\) and \(\mu_k = qf_k\), where

\[
f_k = \frac{C}{C-1} [\delta C^{k-1} + (\mu - 1)].
\]

We remark that the three last sequences are strictly increasing and unbounded. Furthermore \((f_k)\) satisfies the relation \(f_{k+1} = C[f_k - (\mu - 1)]\).

Proof of Proposition 2. \hspace{1cm} 1. We show by induction that for all integer \(k\), \(u_R \in L^{\lambda_k}(\Omega)\) and \(v_R \in L^{\mu_k}(\Omega)\). For \(k = 0\),

\[
\lambda_0 = pf_0 = \frac{pC}{C-1} \left[\frac{\delta}{C} + (\mu - 1)\right] = \frac{p}{p} \left[\frac{p}{N} \mu \tilde{C}^{k_0}\right] = \tilde{\lambda}_0,
\]

and similarly, \(\mu_0 = \tilde{\mu}_0\).

By Lemma 4, \(u_R \in L^{\lambda_0}(\Omega)\) and \(v_R \in L^{\mu_0}(\Omega)\). Suppose that \((u_R, v_R) \in L^{\lambda_k}(\Omega) \times L^{\mu_k}(\Omega)\). First we establish that \(\lambda_k = a_k + p\mu\). By condition (3.1),

\[
1 = \frac{p + a_k}{\lambda_k} + q \frac{\mu - 1}{\mu_k} = \frac{p}{\lambda_k} - \frac{q}{\mu_k} + \frac{a_k}{\lambda_k} + \frac{\mu}{\mu_k},
\]

thus

\[
\frac{a_k}{p f_k} + \frac{\mu}{f_k} = 1
\]

which implies \(a_k = p(f_k - \mu) = \lambda_k - p\mu\), and similarly \(\mu_k = b_k + q\mu = q(f_k - \mu)\). Now when we take \(\pi_p = C\pi\) and \(\pi_q = C\pi\), we then have

\[
\left[1 + \frac{a_k}{p}\right] \pi_p = (1 + f_k - \mu)C\pi = pf_{k+1} = \lambda_{k+1}.
\]

and similarly \([1 + \frac{b_k}{q}]\pi_q = \mu_{k+1}\). Since \((u_R, v_R) \in L^{\lambda_k}(\Omega) \times L^{\mu_k}(\Omega)\), we conclude, according to Lemma 3, that

\[
(u_R, v_R) \in L^{\lambda_{k+1}}(\Omega) \times L^{\mu_{k+1}}(\Omega).
\]

So \(u_R \in L^{\lambda_k}(\Omega)\), and \(v_R \in L^{\mu_k}(\Omega)\), for all integer \(k\).

2. Now we prove that \(u_R\) and \(v_R\) are bounded. By Lemma 3, we have

\[
\|u_R\|_{L^{\lambda_{k+1}}}^{\lambda_{k+1}} \leq K_R \left\{ \theta_p \left[1 + \frac{a_k}{p}\right] \left(C_R|\theta_0||u_R|^{\lambda_k} + \|v_R\|_{\mu_k}^{\mu_k}\right)^{1/p} \right\}^\frac{\lambda_k}{\lambda_{k+1}},
\]

\[
\|v_R\|_{L^{\mu_{k+1}}}^{\mu_{k+1}} \leq K_R \left\{ \theta_q \left[1 + \frac{b_k}{q}\right] \left(C_R|\theta_0||u_R|^{\lambda_k} + \|v_R\|_{\mu_k}^{\mu_k}\right)^{1/q} \right\}^\frac{\mu_k}{\mu_{k+1}}.
\]
We remark that
\[ \frac{\lambda_{k+1}}{1 + \frac{a_k}{p}} = pC \quad \text{and} \quad \frac{\mu_{k+1}}{1 + \frac{b_k}{q}} = qC. \]
Consequently,
\[
\|u_R\|_{\lambda_{k+1}}^p \leq 2^C K_p \theta_p^p C^p \left[ 1 + \frac{a_k}{p} \right] \|m_0 C_R\| C \max \left( \|u_R\|_{\lambda_k}^p, \|v_R\|_{\mu_k}^p \right),
\]
\[
\|u_R\|_{\mu_{k+1}}^q \leq 2^C K_q \theta_q^q C^q \left[ 1 + \frac{b_k}{q} \right] \|m_0 C_R\| C \max \left( \|u_R\|_{\lambda_k}^q, \|v_R\|_{\mu_k}^q \right).
\]
We have
\[ 1 + \frac{a_k}{p} = 1 + \frac{b_k}{q} = 1 + f_k - \mu < \frac{C}{C - 1} \frac{\delta}{C} + \mu - 1 \right] C^k.
\]
Take
\[ A := \frac{C}{C - 1} \frac{\delta}{C} + \mu - 1 \right] [K_p + K_q] \]
and \( \theta := 2|m_0| \max(\theta_p, \theta_q) \), then we can write
\[
\max \left( \|u_R\|_{\lambda_{k+1}}^p, \|v_R\|_{\mu_{k+1}}^q \right) \leq (A^q \theta)^C C^{k+1} C^C \max \left( \|u_R\|_{\lambda_k}^C, \|v_R\|_{\mu_k}^C \right).
\]
We construct an iterative relation
\[ E_{k+1} \leq r_k + C E_k \]
where \( E_k = \ln \max(\|u_R\|_{\lambda_k}, \|v_R\|_{\mu_k}) \), and \( r_k = ak + b \), with \( a = \ln C^C \) and \( b = \ln [A^q \theta C_R] C^C \). Proceeding step by step, we find
\[
E_{k+1} \leq r_k + C r_{k-1} + C^2 r_{k-2} + \ldots + C^k r_0 + C^{k+1} E_0,
\]
\[
E_{k+1} \leq C^{k+1} E_0 + \sum_{i=0}^{k} C^i r_{k-i}.
\]
Let us evaluate
\[ \sigma_k := \sum_{i=0}^{k} C^i r_{k-i}. \]
We have \( r_{k-i} = a(k-i) + b = ak + b - ai \), then
\[ \sigma_k = (ak + b) \sum_{i=0}^{k} C^i - a \sum_{i=0}^{k} i C^i = \frac{b C^{k+2} + (a-b) C^{k+1} + (1-C) ak - [C(a+b)-b]}{(C-1)^2}. \]
Since \( C > 1 \), and \( a, b \) are positive, we have
\[ \sigma_k \leq \frac{b C^{k+2} + (a-b) C^{k+1}}{(C-1)^2}. \]
then
\[ E_{k+1} \leq \frac{bC^{k+2}}{(C-1)^2} + C^{k+1} \left[ \frac{a-b}{(C-1)^2} + E_0 \right]. \]

By an appropriate choice for the constants \( K_p \) and \( K_q \), we ensure that
\[ \frac{b-a}{(C-1)^2} \geq E_0. \]

Recall that
\[ b - a = C \ln \frac{A^q \theta C_R}{C^q} \quad \text{with} \quad A = \frac{C}{C-1} \left( \frac{\delta}{C} + \mu - 1 \right) [K_p + K_q]; \]

hence \( E_{k+1} \leq \frac{bC^{k+2}}{(C-1)^2} \). By the definition of \( E_{k+1} \) and the last inequality, we obtain
\[ \lambda_{k+1} \ln \| u_R \|_{\lambda_{k+1}} \leq E_{k+1} \leq \frac{bC^{k+2}}{(C-1)^2}, \]

thus
\[ \ln \| u_R \|_{\lambda_{k+1}} \leq \frac{bC^{k+2}}{\lambda_{k+1}(C-1)^2}. \]

Letting \( k \to +\infty \), we find
\[ \ln \| u_R \|_{\infty} \leq \frac{bC}{p\delta(C-1)}, \quad \text{or} \quad \ln \| u_R \|_{\infty} \leq \frac{N}{\delta p^2} b. \]

Similarly
\[ \ln \| v_R \|_{\infty} \leq \frac{N}{\delta q^2} b. \]

We deduce the existence of constants \( C_p \) and \( C_q \) such that:
\[ \| u_R \|_{\infty} \leq C_p \quad \text{and} \quad \| v_R \|_{\infty} \leq C_q. \]

Take
\[ C_p = \exp \frac{N}{\delta p^2} b, \quad \text{and} \quad C_q = \exp \frac{N}{\delta q^2} b. \]

Then \( C_p \) and \( C_q \) are greater than 1, which is compatible with the remark noted at the beginning of the proof of Lemma 3. This completes the proof of Proposition 1.

**Proof of Theorem 2.1.** If \( \| u_R \|_p^p + \| v_R \|_q^q < R \), then \((u_R, v_R)\) furnishes a solution of the system (1.1). We have
\[ \| u_R \|_p^p + \| v_R \|_q^q \leq C_p^p + C_q^q \leq 2 \exp \frac{N}{\delta p^2} b; \]

so it is sufficient to have \( 2 \exp \frac{N}{\delta p^2} b < R \) for \( R \) large enough, to get \((u_R, v_R)\) solution of the initial system (1.1). Replacing \( b \) by its expression, we obtain
\[ (A^q \theta C_R)^{\frac{1}{2q}} < \frac{N}{2} \]

i.e.
\[ C_R < \frac{R^{\frac{1}{2p}}} {2^{\frac{1}{2q}} \theta A^q}. \]
But $\delta$ can be chosen such that
\[
\frac{\delta p}{CN} > \frac{N^2}{pq} \mu = \left(\frac{p^* q^*}{(p^* - p)(q^* - q)}\right) \mu
\]
and we can take $C_R < \frac{R^p - p^* - q^*}{2 \frac{\mu}{\theta A}}$. Then $(u_R, v_R)$ is solution of system (1.1) if
\[
C_R = o\left(\frac{R^p - p^* - q^*}{\theta A}\right)
\]
for $R$ sufficiently large.

**Examples**

Now, we present functions satisfying the hypotheses in our main result.

For $1 < \gamma < \min\left(\frac{p}{p^*}, \frac{q}{q^*}\right)$, let
\[
H(u, v) = (u^p + v^q)^\gamma
\]
be defined on $\mathbb{R}^2_+$. Then $H$ satisfies the hypotheses of Theorem 2.1.

For $\alpha, \beta \geq 0$, $\alpha + 1 + \frac{\beta + 1}{p} > 1$ and $\alpha + 1 + \frac{\beta + 1}{q} < 1$, let
\[
H(u, v) = u^{\alpha + 1} v^{\beta + 1}
\]
be defined on $\mathbb{R}^2_+$. Then $H$ satisfies the hypotheses of Theorem 2.1.

**References**


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