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A priori bounds and global existence for a strongly coupled quasilinear parabolic system modeling chemotaxis *

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Abstract

A priori bounds are found for solutions to a strongly coupled reaction-diffusion system that models competition of species in the presence of chemotaxis. These bounds are used to prove the existence of global solutions.

1 Introduction

Chemotaxis is a property of certain living organisms to be repelled or attracted to chemical substances. In a 1970 paper [7] E.F. Keller and L.A. Segel proposed a model to describe the aggregation of the slime mold *Dictyostelium discoidium*. It consists of two strongly coupled diffusion equations,

$$\begin{aligned}\partial_t S &= \gamma_1 \Delta S - \gamma_2 S + \gamma_3 u,\\ \partial_t u &= \Delta u - \nabla\cdot(u\nabla X(S)) \quad \text{in } \Omega,\end{aligned}$$

together with homogeneous Neumann boundary conditions and nonnegative initial conditions. Here S is the concentration of the chemotactic agent and u is the concentration of organisms. In the original model the *sensitivity function* $X(S)$ was taken to be χS for some constant χ. Since, this model has been studied by many authors including Jäger and Luckhaus [6], Herrero and Velázquez [5], Nagai and Senba [16, 17], Gajewski and Zacharias [4]. Global solutions as well as solutions that blow up in finite time have been found. In many, if not most, papers the sensitivity function is assumed to be linear, however other functions such as the logarithm and power functions have also been considered. The model is often simplified by restricting the dimension of Ω, by restriction to consideration of radially symmetric solutions, or by reducing the system to the elliptic-parabolic system that is obtained as the limit by letting the diffusion coefficient γ_1 tend to infinity (see e.g. [6, 15, 17, 18]) The present paper was motivated by recent results of Le and Smith [13] on the existence of steady state

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solutions for a model that incorporates both competition and chemotaxis (for example in a chemostat). Such a model was originally introduced and studied by Lauffenburger and coworkers [8, 11, 12]. The model for microbial competition in a chemostat considered by Le and Smith involves the following quasilinear parabolic system:

$$\begin{aligned}\partial_t S &= d_0 \Delta S - \mu_0 \cdot \nabla S - c_0 S - f_0(x, S, u)\\ \partial_t u_k &= \nabla \cdot (-\chi_k(S) u_k \nabla S) + d_k \Delta u_k - \mu_k \cdot \nabla u_k - c_k u_k + u_k f_k(x, S),\end{aligned}$$

where $x \in \Omega$, a bounded domain in $\mathbb{R}^n$, $\chi_k := X_k'$, $k = 1, 2, \dots, m$, $u = (u_1, u_2, \dots, u_m)$ are the population densities of the competing organisms, and S is the concentration of nutrient which here plays the role of chemotactic agent. If we use $\nu(x)$ to denote the unit outward normal vector at $x \in \partial\Omega$, and $\partial_\nu \equiv \partial/\partial\nu$ to denote the outward normal derivative at the boundary, then the boundary conditions are of the form

$$\begin{aligned} d_0 \frac{\partial S}{\partial \nu} + \zeta(x) S &= \eta(x)\\ d_k \frac{\partial u_k}{\partial \nu} + r_k(x, \frac{\partial S}{\partial \nu}) u_k &= 0 \quad \text{for } k = 1, 2, \dots, m,\end{aligned}$$

where ζ, and r_k are nonnegative functions. The vector fields μ_k represent convection currents in the chemostat. Le and Smith found conditions under which this problem has a positive steady state. In this paper we shall be concerned with the existence of global time-dependent solutions for problems such as this.

2 Statement of the Problem and Preliminaries

As was done in [13], we will assume that the convection currents are gradient fields:

$$\mu_j = \nabla B_j \quad \text{for } j = 0, 1, \dots, m,$$

so that we look at the problem

$$\partial_t S = d_0 \Delta S - \nabla B_0 \cdot \nabla S - F_0(x, t, S, u), \quad \text{in } \Omega, \tag{1}$$

$$\begin{aligned}\partial_t u_k = d_k \Delta u_k &- \nabla \cdot (\chi_k(S) u_k \nabla S) - \nabla B_k \cdot \nabla u_k\\ &- \lambda_k u_k + F_k(x, t, S, u), \quad \text{in } \Omega,\end{aligned} \tag{2}$$

$$d_0 \partial_\nu S + \zeta(x) S = \eta(x) \quad \text{on } \partial\Omega, \tag{3}$$

$$d_k \partial_\nu u_k + \beta_k(x, S, u) u_k - \chi_k(S) u_k \partial_\nu S = 0, \tag{4}$$

for $k = 1, 2, \dots, m,$ on $\partial\Omega$, with initial conditions

$$0 \le S(\cdot, 0) = S_0 \in C^2(\overline{\Omega}), \quad 0 \le u_k(\cdot, 0) = u_{k0} \in C^2(\overline{\Omega}), \quad k = 1, 2, \dots m, \tag{5}$$

that satisfy the given boundary conditions. We will assume that Ω is a bounded domain whose boundary $\partial\Omega$ is of class C^3. The diffusion coefficients d_k, $k =$

$0, 1, \dots, m$, are positive constants and the parameters λ_k, $k = 1, \dots, m$, are real constants. We further assume that for all relevant values of j

$$\begin{gathered} F_j \in C^1(\overline{\Omega} \times \mathbb{R}^{m+2}), \\ 0 < \chi_j \in C^2(\mathbb{R}) \quad \text{and is bounded} \\ B_j \in C^3(\overline{\Omega}), \\ \beta_j \in C^2(\overline{\Omega} \times \mathbb{R}^{m+1}), \text{ and } \eta, \zeta \in C^2(\overline{\Omega}). \end{gathered}$$

We also assume that β_j for $j = 1, 2, \dots, m$, ζ, and F_0 are nonnegative-valued functions, that $F_0(x, t, u, 0) = 0$, and $F_j(x, t, u, S) = 0$ when $u_j = 0$, and that there exists a constant k_F such that

$$F_j(x, t, S, u) \le k_F F_0(x, t, S, u) \tag{6}$$

for $j = 1, 2, \dots, m$. Throughout we will be dealing with nonnegative solutions in the classical sense, or at least with solutions each of whose components belongs to the space $C^1([0, T), H^1(\Omega)) \cap C((0, T), H^2(\Omega))$ for some maximal $T > 0$. See section 5 where we quote the theorem by Amann that ensures positivity of solutions.

We will use $\| \cdot \|_{s,p,\Omega}$ to denote the norm on the Sobolev space $W_p^s(\Omega)$. For $s = k$, a nonnegative integer, this space is the familiar the space of functions that are in $L_p(\Omega)$ together with all of its derivatives of order $\le k$. For positive non-integral order s the spaces $W_p^s(\Omega)$ are usually defined by means of the *real interpolation method*, see for example [1, 14]. This means that for $0 < s < 1$ there exists a constant K_s such that for all $u \in W_p^1(\Omega)$

$$\|w\|_{s,p,\Omega} \le K_s |\ w\|_{0,p,\Omega}^{1-s} \|w\|_{1,p,\Omega}^s.$$

The Sobolev-Slobodeckii spaces $\tilde{W}_p^s(\Omega)$ are defined, as follows. If s is an integer then it coincides with the Sobolev space $W_p^s(\Omega)$. For $0 < s < 1$ define the following seminorm:

$$[w]_{s,p,\Omega} := \left(\int_\Omega \int_\Omega \frac{|w(x) - w(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{1/p}. \tag{7}$$

When s is not an integer the Sobolev-Slobodeckii space $\tilde{W}_p^s(\Omega)$ is defined to be the space of all functions $w \in W_p^{[s]}(\Omega)$ such that

$$\|w\|_{s,p,\Omega} := \left(\|w\|_{[s],p,\Omega}^p + \sum_{|\alpha|=[s]} [\partial^\alpha w]_{s-[s],p,\Omega}^p \right)^{1/p} < \infty. \tag{8}$$

Since we are assuming that $\partial\Omega$ is of class C^3, it has the C^1-*regularity property* [1]. This, in, turn implies that the space $\tilde{W}^{s,p}(\Omega)$ coincides, algebraically and topologically, with $W_p^s(\Omega)$ [1]. By employing partitions of unity one can similarly define Sobolev and Sobolev-Slobodeckii spaces on $\partial\Omega$.

The notation C^{k-} will be used to denote functions whose derivatives of order $k-1$ are Lipschitz continuous. When context precludes confusion we will also use the notation $W_p^s(\Omega)$ for vector-valued functions.

The following theorem is a simplified version of Theorem 5.4 below.

Theorem 2.1 *Suppose that there exists a continuous function A and a positive number Q such that for all $j = 0, 1, \ldots, m$*

$$F_j(x, t, S, u), \leq A(S)[1 + |u|^Q],$$

Then problem (1)-(5) has a classical solution on $[0, \infty)$ provided that for some $0 < \delta < 1$, $\|\eta\|_{C^{1+\delta}(\Omega)}$ and $\|S_0\|_\infty$ are sufficiently small.

3 A priori $L_q(\Omega)$ bounds on u.

Since the following proofs for the case $m > 1$ are nearly the same as those for the case $m = 1$, we simplify matters by assuming that $m = 1$ and by dropping unnecessary subscripts. Later, we will outline the modifications that are needed to handle the case $m > 1$.

Let $[0, T)$ denote the maximal interval on which problem (1)-(5) has a solution. We let S_* denote the so-called *wash-out solution*, a steady state corresponding to the situation where there is a total absence of organisms. We assume such a solution exists:

$$\begin{aligned} -d_0 \Delta S_* + \nabla B_0 \cdot \nabla S_* = 0 \quad \text{in } \Omega, \\ d_0 \partial_\nu S_* + \zeta S_* = \eta \quad \text{on } \partial\Omega. \end{aligned}$$

From the maximum principle it follows that $S(x, t) \geq 0$ and $S_*(x, t) \geq 0$. We know from standard regularity results [9] that $S_* \in C^{3-}(\overline{\Omega})$. Define $s := S - S_*$, so that

$$\partial_t s = d_0 \Delta s - \nabla B_0 \cdot \nabla s - F_0(x, t, S, u) \quad \text{in } \Omega, \quad \partial_\nu s + \zeta s = 0, \quad \text{on } \partial\Omega. \quad (9)$$

From the maximum principle we may deduce that $s(x, t) \leq \overline{s} := \|S_0 - S_*\|_\infty$. If we define $\underline{s} := -\|S_*\|_\infty$ then

$$\underline{s} \leq s(x, t) \leq \overline{s} \quad \forall x \in \Omega \text{ and } t > 0. \quad (10)$$

We will now try to find a Lyapunov function of the form

$$L(t) := \int_\Omega \exp(-B_0(x)/d_0)\, \mathcal{L}(s(x, t), D(x)u(x, t))\, dx,$$

where $\mathcal{L}$ has the form

$$\mathcal{L}(s, v) = (v^q + K) \exp(z(s)),$$

with $q \geq 1$ and $K > 0$ constant, $D(x) := \exp(B_0(x)/d_0 - B(x)/d)$, and z a strictly increasing, convex function yet to be determined. At times we will use v to denote Du. It is convenient to rewrite the equations for s and u as

$$\begin{aligned} \partial_t s &= \exp(B_0/d_0)\nabla\cdot\exp(-B_0/d_0)d_0\nabla s - F_0(x,t,S,u) \\ \partial_t u &= \exp(B/d)\nabla\cdot\exp(-B/d)(d\nabla u - \chi(S)u\nabla s) \\ &\quad -\chi(S)u\nabla B\cdot\nabla s/d - \nabla\cdot\chi(S)u\nabla S_* - \lambda u + F(x,t,S,u) \end{aligned} \tag{11}$$

We will try to find a function z such that $dL/dt \leq 0$ for λ sufficiently large. To begin with we assume that $\zeta = 0$. A straightforward computation then shows that

$$\begin{aligned} \frac{dL}{dt} &= \int_\Omega \mathcal{L}_s(s,v)\left\{\nabla\cdot\exp(-B_0/d_0)d_0\nabla s - F_0(x,t,S,u)\exp(-B_0/d_0)\right\}dx \\ &\quad + \int_\Omega \mathcal{L}_v\Big\{\nabla\cdot\exp(-B/d)\left[d\nabla u - u\chi(S)\nabla s\right] \\ &\quad - \exp(-B/d)\chi(S)u\nabla B\cdot\nabla s/d - \exp(-B/d)\nabla\cdot\chi(S)u\nabla S_* \\ &\quad -\lambda\exp(-B/d)u + F(x,t,S,u)\exp(-B/d)\Big\}\,dx \\ &= \int_{\partial\Omega}\Big\{\mathcal{L}_s\exp(-B_0/d_0)d_0\partial_\nu s + \mathcal{L}_v\exp(-B/d) \\ &\quad \times\big(d\partial_\nu u - u\chi(S)\partial_\nu s - \chi(S)u\partial_\nu S_*\big)\Big\}\,d\sigma \\ &\quad - \int_\Omega \exp(-B_0/d_0)\Big\{\mathcal{L}_{ss}d_0|\nabla s|^2 + \mathcal{L}_{sv}Dd_0\nabla s\cdot\nabla u - \mathcal{L}_{sv}d_0u\nabla s\cdot\nabla D \\ &\quad +\mathcal{L}_{vs}\exp(-B/d)\left[d\nabla u - u\chi(S)\nabla s\right]\cdot\nabla s \\ &\quad +\mathcal{L}_{vv}\exp(-B/d)\left[d\nabla u - u\chi(S)\nabla s\right]\cdot\left[D\nabla u + u\nabla D\right]\Big\}\,dx \\ &\quad + \int_\Omega \exp(-B/d)\Big\{\mathcal{L}_{vs}\chi(S)u\nabla S_*\cdot\nabla s + \mathcal{L}_{vv}\chi(S)u\nabla S_*\cdot\left[D\nabla u + u\nabla D\right] \\ &\quad -\mathcal{L}_v\chi(S)u\nabla S_*\cdot\nabla B/d\Big\}\,dx + \int_\Omega\Big\{-\mathcal{L}_sF_0(x,t,S,u)\exp(-B_0/d_0) \\ &\quad +\mathcal{L}_v\exp(-B/d)\left[-\lambda u + F(x,t,S,u) - \chi(S)u\nabla B\cdot\nabla S/d\right]\Big\}\,dx \\ &= -\int_{\partial\Omega}\mathcal{L}_v\exp(-B/d)\beta\,d\sigma - \int_\Omega \exp(-B/d)\mathcal{L}_v\chi(S)u\nabla s\cdot\nabla B/d\,dx \\ &\quad - \int_\Omega\Big\{\exp(-B_0/d_0)\left[\mathcal{L}_{ss}d_0 - D\mathcal{L}_{vs}u\chi(S)\right]|\nabla s|^2 + \exp(-B/d) \\ &\quad \times\left[\mathcal{L}_{sv}(d_0+d) - Du\chi(S)\mathcal{L}_{vv}\right]\nabla u\cdot\nabla s + \exp(-B/d)D\mathcal{L}_{vv}d|\nabla u|^2 \\ &\quad + \exp(-B/d)\mathcal{L}_{vv}u^2\chi\nabla s\cdot\nabla D + \exp(-B/d)\mathcal{L}_{vv}u^2\chi\nabla S_*\cdot\nabla D\Big\}dx \\ &\quad - \int_\Omega \exp(-B/d)\Big[\mathcal{L}_{sv}\left(d_0D^{-1}u\nabla s\cdot\nabla D + \nabla s\cdot\nabla S_*u\chi(S)\right) \\ &\quad +D\mathcal{L}_{vv}\nabla u\cdot\nabla S_*u\chi(S) - d\mathcal{L}_{vv}u\nabla u\cdot\nabla D\Big]\,dx \end{aligned}$$

$$+\int_\Omega \exp(-B/d)\Big[\mathcal{L}_v F(x,t,S,u) - \mathcal{L}_s F_0(x,t,S,u)/D$$
$$+\mathcal{L}_v\chi(S)u\nabla S_*\cdot\nabla B/d - \mathcal{L}_v\lambda u\Big]\,dx$$

The surface integral is non-positive. Let $k_0 := \sup\{D(x)\,|x\in\Omega\}$. If we choose the function z such that for some $\delta>0$ we have

$$z'(s)\ge\delta,$$

then we may choose the constant K in the definition of $\mathcal{L}$ large enough that

$$\mathcal{L}_s \ge k_F k_0 \mathcal{L}_v.$$

Hence, if we choose $\lambda \ge \sup\{\nabla S_*(x)\cdot\nabla B_0(x)\|\chi\|_\infty/d\,|x\in\Omega\}$, then the last integral above above will be negative. Next we look at the terms involving quadratic terms in the gradients. We will need the following inequality satisfied for some positive number ϵ:

$$\begin{aligned}&\big(\mathcal{L}_{ss}d_0 - \mathcal{L}_{vs}uD\chi(S)\big)|\nabla s|^2\\ &+D\big(\mathcal{L}_{sv}(d_0+d) - uD\chi(S)\mathcal{L}_{vv}\big)\nabla u\cdot\nabla s + D^2\mathcal{L}_{vv}d|\nabla u|^2\\ &\quad\ge\quad \epsilon\mathcal{L}_{ss}|\nabla s|^2 + \epsilon\mathcal{L}_{vv}D^2|\nabla u|^2.\end{aligned}\tag{12}$$

This will be true provided

$$(\mathcal{L}_{sv}(d_0+d) - uD\chi(S)\mathcal{L}_{vv})^2 \le 4\mathcal{L}_{vv}(d-\epsilon)\left(\mathcal{L}_{ss}(d_0-\epsilon) - \mathcal{L}_{vs}uD\chi(S)\right),$$

That is to say,

$$\begin{aligned}&[z'(s)q(d_0+d) - \chi(S)q(q-1)]^2\\ &\quad\le\quad 4q(q-1)(d-\epsilon)\big[\big(z''(s)+z'(s)^2\big)(d_0-\epsilon) - qz'(s)\chi(S)\big].\end{aligned}\tag{13}$$

This provides us with the following second order differential inequality that needs to be satisfied:

$$z''(s) \ge A_q z'(s)^2 - B_q\chi(S)z'(s) + C_q\chi(S)^2,\tag{14}$$

where

$$\begin{aligned}A_q &:= \frac{q(d+d_0)^2 - 4(q-1)(d-\epsilon)(d_0-\epsilon)}{4(q-1)(d-\epsilon)(d_0-\epsilon)},\\ B_q &:= \frac{q(d_0-d+2\epsilon)}{2(d-\epsilon)(d_0-\epsilon)},\\ C_q &:= \frac{q(q-1)}{4(d-\epsilon)(d_0-\epsilon)}.\end{aligned}$$

To see that A_q is positive (for sufficiently small ϵ) we rewrite it as

$$A_q = (q-1)^{-1}\left[1 + \frac{q(d_0-d)^2 - 4q\epsilon(d_0+d-\epsilon)}{4(d-\epsilon)(d_0-\epsilon)}\right].$$

Although in chemotaxis models it is usually assumed that the function $\chi(S)$ is positive and decreasing (and of course, defined for $S \geq 0$) we will not need to assume such monotonicity. However, we shall assume that $\chi(S)$ is defined for all real S. If necessary, we simply extend χ to the negative real line by setting $\chi(S) = \chi(0)$ for $S < 0$.

Definition. Let

$$\chi_*(s) := \inf\{\chi\big(S_*(x) + s\big) \mid x \in \Omega\}$$
$$\chi^*(s) := \sup\{\chi\big(S_*(x) + s\big) \mid x \in \Omega\}$$

Then

$$\chi_*(s(x,t)) \leq \chi(S(x,t)) \leq \chi^*(s(x,t)).$$

We let z be the solution to the Riccati equation

$$z''(s) = A_q z'(s)^2 - B_q \chi_*(s) z'(s) + C_q \chi^*(s)^2 \tag{15}$$
$$z(\underline{s}) = 0, \quad z'(\underline{s}) = \delta \tag{16}$$

If we set $\epsilon = 0$ then the discriminant

$$D_q := B_q^2 \chi_*^2 - 4A_q C_q \chi^{*2} \leq -q\chi^{*2}/(dd_0),$$

Therefore, we see that $D_q < 0$ provided ϵ is chosen sufficiently small. In that case the right hand side of the above Riccati equation will be positive, so that z will indeed be convex and

$$z'(s) > \delta \text{ for all } s > \underline{s}.$$

Therefore $z(s)$ will exist on some maximum interval $[\underline{s}, s^*]$. If $\overline{s} \leq s^*$ then $L(t)$ will be a Lyapunov function provided the remaining terms in the integrands for expression for dL/dt (i.e. the terms that are of degree 1 in the gradients) add up to something that can be bounded by

$$\epsilon \mathcal{L}_{ss} |\nabla s|^2 + \epsilon \mathcal{L}_{vv} |\nabla u|^2 + C_\epsilon \mathcal{L}_v u \tag{17}$$

for some constant C_ϵ, and that we take λ sufficiently large. To prove this we first show that the following ratios are uniformly bounded:

$$\frac{v^2 \mathcal{L}_{vv}(s,v)^2}{\mathcal{L}_{vv}(s,v)\, v\, \mathcal{L}_v(s,v)}, \quad \frac{v^2 \mathcal{L}_{vs}(s,v)^2}{\mathcal{L}_{ss}(s,v)\, v\, \mathcal{L}_v(s,v)},$$
$$\frac{v^2 \mathcal{L}_{v}(s,v)^2}{\mathcal{L}_{ss}(s,v)\, v\, \mathcal{L}_v(s,v)}, \quad \frac{v^4 \mathcal{L}_{vv}(s,v)^2}{\mathcal{L}_{ss}(s,v) v \mathcal{L}_v(s,v)}.$$

Evaluating these we see that these ratios are, respectively,

$$q-1, \quad \frac{qv^q (z')^2}{(K+v^q)(z''+(z')^2)} < q,$$
$$\frac{qv^q}{(K+v^q)(z''+(z')^2)} < \frac{q}{z''+(z')^2}, \quad \frac{q(q-1)^2}{z''+(z')^2}.$$

Since $z' \geq \delta$ the last two ratios are also uniformly bounded. But because the value for δ has not yet been chosen, it is desirable to show that the bounds may be picked independently of δ. From equation (15) we see that

$$\begin{aligned} z'' + (z')^2 &= \Big[\sqrt{A_q+1}z' - \frac{B_q\chi_*}{2\sqrt{A_q+1}}\Big]^2 + \Big[C_q - \frac{B_q^2}{4(A_q+1)}\Big]\chi_*^2 \\ &\quad + C_q(\chi^{*2} - \chi_*^2) \\ &\geq \Big[C_q - \frac{B_q^2}{4(A_q+1)}\Big] \inf\big\{\chi(s)^2 \mid 0 \leq s \leq \overline{s} + \|S_*\|_\infty\big\} > 0. \end{aligned}$$

For any positive numbers ϵ, α, β, and γ we have for all real values r

$$\alpha r \leq \epsilon\beta r^2 + \Big(\frac{\alpha^2}{4\beta\gamma\epsilon}\Big)\gamma.$$

Setting $v = uD$, we see that there is a positive number K such that

$$\mathcal{L}_v(s,v)v|\nabla s|, \mathcal{L}_{vs}(s,v)v|\nabla s|, \mathcal{L}_{vv}(s,v)v^2|\nabla s|^2 \leq (\epsilon/3)\mathcal{L}_{ss}|\nabla s|^2 + K\mathcal{L}_v u,$$

and

$$\mathcal{L}_{vv}v|\nabla u| \leq \epsilon\mathcal{L}_{vv}|\nabla u|^2 + K\mathcal{L}_v u.$$

Also we have

$$\mathcal{L}_{vv}(s,v)v^2 \leq (q-1)\mathcal{L}_v(s,v)v.$$

Hence all terms that are first of first order in ∇s and ∇u can be controlled and therefore $dL/dt \leq 0$ provided λ is sufficiently large and $s(x,t)$ remains in the interval of existence of $z(s)$. In that case we have a global bound on the $L_q(\Omega)$ norm:

$$\|u(\cdot,t)\|_q \leq (K|\Omega|)^{1/q} + \|u(x,0)\|_q)\exp(z(\overline{s})/q). \tag{18}$$

Since we will need $z(s)$ to be defined on the interval $[\underline{s},\overline{s}]$ we estimate the interval on which the solution $z(s)$ exists. First define

$$Z_\delta(s) := z'(s) - \frac{B_q\chi_*(s)}{2A_q},$$

where the derivative may have to be interpreted in the weak or almost everywhere sense. Then

$$Z'_\delta = A_q Z_\delta^2 + \Bigg[C_q\chi^{*2} - \frac{B_q^2\chi_*^2}{4A_q} - \frac{B_q\chi'_*}{2A_q}\Bigg],$$

$$Z_\delta(\underline{s}) = \delta - \frac{B_q\chi_*(\underline{s})}{2A_q}.$$

We define

$$E_q := \sup\Big\{\big[C_q\chi^{*2} - \frac{B_q^2\chi_*^2}{4A_q} - \frac{B_q\chi'_*}{2A_q}\big] \;:\; \underline{s} \leq s \leq \overline{s}\Big\}.$$

Then $Z_\delta(s) \le Z(s)$ where

$$Z' = A_q Z^2 + E_q, \quad Z(\underline{s}) = \delta - \frac{B_q \chi_*(\underline{s})}{2A_q}.$$

The solution Z will exist on some maximal interval of the form $[\underline{s}, s^*)$. We have three cases depending on the sign of E_q. If $E_q > 0$ then one has as solution

$$Z(s) = \sqrt{E_q/A_q}\tan\left(\sqrt{A_q E_q}(s + s_0)\right)$$

where s_0 must be chosen so that

$$\sqrt{E_q/A_q}\tan(\sqrt{A_q E_q}(\underline{s} + s_0)) = \delta.$$

Suppose that

$$\overline{s} - \underline{s} < \frac{\pi}{2\sqrt{A_q E_q}}.$$

Since E_q depends on $\overline{s}$ and $\underline{s}$ this assumption might be more appropriately written as

$$(\overline{s} - \underline{s})\sqrt{E_q} < \frac{\pi}{2\sqrt{A_q}}. \tag{19}$$

There exists a number $0 < \delta_1 < \pi/2$ such that $(\overline{s} - \underline{s})\sqrt{E_q} < (\pi - 2\delta_1)/(2\sqrt{A_q})$. Now choose

$$\delta = \sqrt{E_q/A_q}\tan(\delta_1/2), \quad s_0 = \delta_1/(2\sqrt{A_q E_q}) - \underline{s},$$

so that Z satisfies the initial condition and $\sqrt{E_q/A_q}(\overline{s} + s_0) < \pi/2 - \delta_1/2$. Therefore Z (and hence also Z_δ) is defined on the entire interval $[\underline{s}, \overline{s}]$. This means that if (S, u) is a solution of problem (1)-(5) on $[0, T)$ such that (19) is satisfied, then the $L_q(\Omega)$ norm of $u(\cdot, t)$ is uniformly bounded on this interval. If $E_q = 0$ we have the solution $Z(s) = \delta[1 - \delta A_q(s - \underline{s})]^{-1}$, so that the solution Z is defined on $[\underline{s}, \overline{s}]$ provided that

$$\overline{s} - \underline{s} < s^* - \underline{s} = \frac{1}{\delta A_q}.$$

We may choose δ small enough so that the above inequality is satisfied irrespective of the initial and boundary conditions on S and u. If $E_q < 0$ all solutions with initial condition $Z(\underline{s}) < \sqrt{-E_q/A_q}$ exist for all $s > \underline{s}$, and hence we have $s^* = \infty$ provided we choose $\delta < \sqrt{-E_q/A_q}$. We have proved the following theorem in case $m = 1$ and $\zeta = 0$.

Theorem 3.1 *If $E_q \le 0$ then $\|u(\cdot, t)\|_q$ is uniformly bounded on $[0, T)$ provided λ is sufficiently large. If $E_q > 0$ then $\|u(\cdot, t)\|_q$ is bounded on $[0, T)$ provided λ is sufficiently large and provided $\|S_0\|_\infty$ and $\|\eta\|_{C^{1+a}(\Omega)}$ are sufficiently small.*

Remark. When one models chemotaxis it is usually assumed that χ is a non-increasing function. Considering the fact that the size of the nutrient is smaller than the size of the individuals feeding on it, it is reasonable to expect $d_0 > d$. Since ϵ was picked so as to ensure that $4A_qC_q > B_q^2$, it follows that in this situation $E_q > 0$.

Remark. In case $d_0 > d$ and χ is a positive, non-increasing, convex function then

$$E_q \leq C_q\chi(0)^2 - B_q\chi'(0)/(2A_q),$$

which has the virtue of being independent of S_*.

To take care of the situation where $\zeta \neq 0$, we make the change of variables

$$\tilde{s} := (s - \underline{s})\exp(Z(x) - \mu t), \quad \tilde{u} := u\exp(-\mu t),$$

where Z is chosen such that $\partial_\nu Z(x) = \zeta(x)$ on $\partial\Omega$. Since the boundary is of class C^3 it is easily verified that we can find such a function $Z \in C^2(\overline{\Omega})$. Straightforward computation then leads to the equations

$$\begin{gathered}
\partial_t\tilde{s} = d_0\Delta\tilde{s} - \nabla\tilde{B}_0\cdot\nabla\tilde{s} - \tilde{F}_0(x,t,S,u),\\
\partial_t\tilde{u} = d_k\Delta\tilde{u} - \nabla\cdot(\chi_k(S)\tilde{u}\nabla S) - \nabla B_k\cdot\nabla\tilde{u} - (\lambda+\mu)\tilde{u} + \tilde{F}(x,t,S,u),
\end{gathered}$$

where

$$\begin{gathered}
\tilde{B}_0 := B_0 + 2d_0Z,\\
\tilde{F}_0(x,t,S,u) = \big[-d_0\nabla Z\cdot\nabla Z + d_0\Delta Z - \big(\nabla B_0\cdot\nabla Z\big) + \mu\big]\tilde{s} + \exp(Z-\mu t)F_0(x,t,S,u),\\
\tilde{F}(x,t,S,u) = \exp(-\mu t)F(x,t,S,u).
\end{gathered}$$

Choosing μ large enough ensures that $\tilde{F}_0 \geq 0$ and one easily sees that $\tilde{F}$ and $\tilde{F}_0$ satisfy the same restrictions that were put on F and F_0. The above proof therefore applies equally well to the system of equations for $\tilde{s}$ and $\tilde{u}$.

When $m > 1$ we use $\mathcal{L}(s,v) := (mK + |v|^q)\exp z(s)$. Instead of condition (12) we now need to show nonnegativity of the quadratic form induced by the matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma^T & \mathcal{D} \end{pmatrix}$$

where

$$\begin{gathered}
\alpha := (d_0-\epsilon)(z'^2 + z'')(mK + |v|^q) - \textstyle\sum_{j=1}^m qz'\chi_j v_j^q,\\
\beta := (qz'd_1v_1^{q-1}, qz'd_2v_2^{q-1}, \dots, qz'd_mv_m^{q-1}),\\
\gamma := \Big(qz'd_0v_1^{q-1} - q(q-1)\chi_1v_1^{q-1}, qz'd_0v_2^{q-1} - q(q-1)\chi_2v_2^{q-1},\\
\dots, qz'd_0v_m^{q-1} - q(q-1)\chi_mv_m^{q-1}\Big),\\
\mathcal{D} := \operatorname{diag}\big(q(q-1)(d_i-\epsilon)v_i^{q-2}\big).
\end{gathered}$$

But such a quadratic form is simply the sum of m quadratic forms induced by the 2×2 matrices

$$\begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i^T & \mathcal{D}_i \end{pmatrix}, \quad i = 1, 2, \dots, m,$$

and each of these leads, as before, to an inequality of the form (13) with d replaced by d_i and χ replaced by χ_i. This, in turn, leads to inequalities of the form (14):

$$z''(s) \geq A_q^{(i)} z'(s)^2 - B_q^{(i)} \chi_i(S) z'(s) + C_q^{(i)} \chi_i(S)^2,$$

We define $\chi_*^{(i)}$, $\chi^{(i)*}$, $E_q^{(i)}$, and $D_q^{(i)}$ as before, and

$$A_q := \max_i A_q^{(i)}, \quad E_q := \max_i E_q^{(i)},$$
$$\chi^* := \max_i \chi^{(i)*}, \quad \chi_{i*} := \min_i \chi_*^{(i)}.$$

The rest of the proof proceeds as in the case $m = 1$.

The last condition in the statement of the theorem is needed to control the sizes of $\underline{s} = -\|S_*\|_\infty$ and $\overline{s} = \|S_0 - S_*\|_\infty$ so that $\overline{s} - \underline{s}$ can be made sufficiently small. Of course, when $E_q \leq 0$ then, irrespective of the value of λ, solutions grow at most exponentially in the $L_q(\Omega)$ norm for any $1 \leq q < \infty$.

When there are no convection currents and we have homogeneous boundary conditions, then many of the terms in the expression for dL/dt are zero so that the statement of theorem 3.1 is true for $\lambda = 0$:

Corollary 3.2 *Suppose that $B_i \equiv 0$ for $i = 0, 1, \dots, m$ and that ζ and η are constant functions. Then the conclusions of Theorem 3.1 hold with $\lambda = 0$.*

Other Lyapunov functions have been found for systems modelling chemotaxis [6, 20]. However, the reaction terms in our problem make it quite different from those studied before, so that there does not seem to be a way to compare those Lyapunov functions with the one used in this paper.

4 A priori bounds on ∇S

Our next objective is to obtain $L_2(\Omega)$ bounds on $\nabla S(\cdot, t)$. For this we will use the so-called *Uniform Gronwall Inequality* [20].

Lemma 4.1 *Let g, h, and y be three positive, locally integrable functions on $[t_0, \infty)$ such that y' is locally integrable on $[t_0, \infty)$, and which satisfy*

$$\frac{dy}{dt} \leq gy + h \text{ for } t \geq t_0,$$
$$\int_t^{t+r} g(s)\, ds \leq a_1, \quad \int_t^{t+r} h(s)\, ds \leq a_2 \quad \int_t^{t+r} y(s)\, ds \leq a_3 \quad \text{for } t \geq t_0,$$

where r, a_1, a_2, and a_3 are positive constants. Then

$$y(t + r) \leq \Big(\frac{a_3}{r} + a_2\Big) \exp(a_1), \quad \forall t \geq t_0.$$

We shall apply this lemma to $y(t) := \int_\Omega |\nabla S(x,t)|^2\,dx$.

Lemma 4.2 *Suppose that $F_0(\cdot,t,S(\cdot,t),u(\cdot,t))$ is uniformly bounded in $L_2(\Omega)$ for all $0 \le t \le t^*$. Then there exist constants c_1 and c_2 such that for all $0 \le t < t^* - r$ and $r \ge 0$*

$$\int_t^{t+r}\int_\Omega |\nabla S(x,s)|^2\,dx\,ds \le c_1 + c_2 r. \tag{20}$$

Proof. Multiplying the equation for S by S, integrating over the cylinder $\Omega \times (t, t+r)$ and doing the typical integration by parts we obtain

$$\begin{aligned}
&\frac{1}{2}\Big[\int_\Omega S(x,t+r)^2\,dx - \int_\Omega S(x,t)^2\Big]\\
&\le \int_t^{t+r}\int_\Omega \Big[-d_0|\nabla S|^2 - \frac{1}{2}\nabla\cdot(S^2\nabla B_0) + \frac{1}{2}S^2\Delta B_0\Big]\,dx\,ds\\
&\quad + \int_t^{t+r}\int_{\partial\Omega} d_0 S\frac{\partial S}{\partial\nu}\,d\sigma\,ds\\
&\le \int_t^{t+r}\int_\Omega [-d_0|\nabla S|^2 + cS^2]\,dx\,ds + \int_t^{t+r}\int_{\partial\Omega}\Big(d_0 S\frac{\partial S}{\partial\nu} - \frac{1}{2}\frac{\partial B_0}{\partial\nu}S^2\Big)\,d\sigma\,ds,
\end{aligned}$$

for some positive constant c. Using the boundary condition we obtain

$$\begin{aligned}
\int_t^{t+r}\int_\Omega d_0|\nabla S|^2\,dx\,ds \quad &\le \quad \frac{1}{2}\Big[\int_\Omega S(x,t)^2\,dx - \int_\Omega S(x,t+r)^2\,dx\Big]\\
&\quad + \int_t^{t+r}\int_\Omega cS^2\,dx\,ds\\
&\quad + \int_t^{t+r}\int_{\partial\Omega}\Big(\frac{1}{2}\Big|\frac{\partial B_0}{\partial\nu}\Big|S^2 + |\eta|S\Big)\,d\sigma\,ds.
\end{aligned}$$

Since S is bounded there are constants c_1 and c_2 so that the right the right-hand side can be bounded by $d_0(c_1 + c_2 r)$. This concludes the proof.

To obtain a differential inequality for $y(t)$ we multiply the equation for S by ΔS and integrate by parts over Ω:

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |\nabla S|^2\,dx \quad &= \quad \int_{\partial\Omega} S_t\frac{\partial S}{\partial\nu}\,d\sigma - d_0\int_\Omega(\Delta S)^2\,dx\\
&\quad + \int_\Omega[(\nabla B_0\cdot\nabla S)\Delta S + F_0\Delta S]\,dx.
\end{aligned} \tag{21}$$

We can bound the third term on the right hand side using the boundedness of ∇B_0 and Young's inequality:

$$\int_\Omega(\nabla B_0\cdot\nabla S)\Delta S\,dx \le \epsilon_1\int_\Omega(\Delta S)^2\,dx + C(\epsilon_1)\int_\Omega(\nabla S)^2\,dx.$$

Note that by Young's inequality we have for arbitrarily small ϵ_2

$$-\int_\Omega S\Delta S\,dx \le \epsilon_2 \int_\Omega (\Delta S)^2\,dx + C(\epsilon_2)\int_\Omega S^2\,dx,$$

while on the other hand

$$-\int_\Omega S\Delta S\,dx = \int_\Omega |\nabla S|^2\,dx - \int_{\partial\Omega} S\frac{\partial S}{\partial\nu}\,d\sigma \ge \int_\Omega |\nabla S|^2\,dx - d_0^{-1}\int_{\partial\Omega}(\eta^2+S^2)\,d\sigma.$$

Putting these inequalities together we have

$$\int_\Omega |\nabla S|^2\,dx \le \epsilon_2\int_\Omega (\Delta S)^2\,dx + C(\epsilon_2)\int_\Omega S^2\,dx + d_0^{-1}\int_{\partial\Omega}(\eta^2+S^2)\,d\sigma. \tag{22}$$

Hence, with application of Young's inequality to the integral $F_0\Delta S$, equation (27) yields the inequality

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |\nabla S|^2\,dx \\
&\le \int_{\partial\Omega} S_t\frac{\partial S}{\partial\nu}\,d\sigma - (d_0-\epsilon_1-\epsilon_2 C(\epsilon_1)-\epsilon_3)\int_\Omega (\Delta S)^2\,dx \\
&\quad + \int_\Omega [C(\epsilon_1)C(\epsilon_2)]S^2\,dx + C(\epsilon_1)d_0^{-1}\int_{\partial\Omega}(\eta^2+S^2)\,d\sigma + C(\epsilon_3)\int_\Omega F_0^2\,dx.
\end{aligned} \tag{23}$$

The surface integral has the required property for applying the uniform Gronwall lemma:

$$\int_t^{t+r}\int_{\partial\Omega} S_t\frac{\partial S}{\partial\nu}\,d\sigma\,ds = d_0^{-1}\int_{\partial\Omega}(\eta S-\zeta S^2/2)\,d\sigma\Big|_t^{t+r} \le C_1(r).$$

We define $\tilde h(t)$ to be the sum of all terms on the right-hand side except the integral of $(\Delta S)^2$. By choosing ϵ_1, ϵ_2 and ϵ_3 sufficiently small we have

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |\nabla S|^2\,dx \le -d_0/2\int_\Omega (\Delta S)^2\,dx + \tilde h(t), \tag{24}$$

where $\int_t^{t+r}\tilde h(s)\,ds \le C_2(r)$. Another application of inequality (22) allows us to write

$$\frac{\partial}{\partial t}\int_\Omega |\nabla S|^2\,dx \le -d_0/\epsilon_2\int_\Omega (\nabla S)^2\,dx + h(t), \tag{25}$$

where

$$h(t) := 2\tilde h(t) + d_0 C(\epsilon_2)/\epsilon_2\int_\Omega S^2\,dx + 1/\epsilon_2\int_{\partial\Omega}(\eta^2+S^2)\,d\sigma.$$

Clearly there is a positive function $C_3(r)$ such that

$$\int_t^{t+r} h(t)\,dt \le C_3(r),$$

and by Lemma 4.2 the function $y(t)$ satisfies a similar inequality. Therefore, by the uniform Gronwall lemma we have

Lemma 4.3 *If $\|F_0(\cdot,t,S(\cdot,t),u(\cdot,t))\|_2$ is bounded on the interval $[0,t^*)$ then there is a constant K_S such that*

$$\int_\Omega |\nabla S(x,t)|^2\,dx \le K_S \quad \forall t\in[0,t^*). \tag{26}$$

Combining this with theorem 3.1 we have

Theorem 4.4 *Suppose that there exists a continuous function A such that*

$$|F_0(x,t,S,u)| \le A(S)[1+|u|^{q/2}]. \tag{27}$$

Let $[0,T)$ denote the maximal interval on the positive real axis for which problem (1)-(5) has a classical solution. If $E_q\le 0$ then $\|\nabla S(\cdot,t)\|_2$ is uniformly bounded on $[0,T)$ provided λ is sufficiently large. If $E_q>0$ then $\|\nabla S(\cdot,t)\|_2$ is bounded on $[0,T)$ provided λ is sufficiently large and, for some $0<\delta<1$, $\|\eta\|_{C^{1+\delta}(\Omega)}$ and $\|S_0\|_\infty$ are sufficiently small.

5 Existence of Solutions

We shall make use of the general existence theory developed by H. Amann [2, 3]. The first one of these is very short but presents the essentials. For a complete presentation the reader should consult [3], especially section 14. For $1\le i,j\le m$ we assume

$$a_{ij},a_i,b_i\in C^{2-}(\overline{\Omega}\times\mathbb{R}^m,\mathbb{R}^{m\times m}),\quad a_0,c\in C^{1-}(\overline{\Omega}\times\mathbb{R}^m,\mathbb{R}^{m\times m}).$$

Using the summation convention we define the following elliptic operator and boundary operator

$$\mathcal{A}(\eta)u := -\partial_j(a_{jk}(\cdot,\eta)\partial_k u + a_j(\cdot,\eta)u) + b_j(\cdot,\eta)\partial_j u + a_0(\cdot,\eta)u,$$

and

$$\mathcal{B}(\eta)u := \nu^j\gamma_0(a_{jk}(\cdot,\eta)\partial_k u + a_j(\cdot,\eta)u) + c(\cdot,\eta)\gamma_0 u,$$

interpreted in the sense of traces. Their formal adjoints are

$$\mathcal{A}^{\#}(\eta)u := -\partial_j(a^{\#}_{jk}(\cdot,\eta)\partial_k u + a^{\#}_j(\cdot,\eta)u) + b^{\#}_j(\cdot,\eta)\partial_j u + a^{\#}_0(\cdot,\eta)u,$$

and

$$\mathcal{B}^{\#}(\eta)u := \nu^j\gamma_0(a^{\#}_{jk}(\cdot,\eta)\partial_k u + a^{\#}_j(\cdot,\eta)u) + c^{\#}(\cdot,\eta)\gamma_0 u,$$

where, letting the left superscript ^{t} denote transpose,

$$a^{\#}_{jk} :={}^t a_{kj}\,,\quad a^{\#}_j :={}^t b_j\,,\quad b^{\#}_j :={}^t a_j\,,\quad a^{\#}_0 :={}^t a_0\,,\quad c^{\#} :={}^t c.$$

Let a_π and b_π denote the principal symbols for $\mathcal{A}$ and $\mathcal{B}$:

$$a_\pi(x,\eta,\xi) := a_{ij}(x,\eta)\xi^i\xi^j,\quad\text{and}\quad b_\pi(x,\eta,\xi) := \nu^i a_{ij}(x,\eta)\xi^j\,,$$

where $\xi = (\xi^1, \xi^2, \dots, \xi^n) \in \mathbb{R}^n$. We assume that for each η, the operator $\mathcal{A}(\eta)$ is *normally elliptic*. By this is meant that for each $x \in \overline{\Omega}$, $\eta \in \mathbb{R}^m$, and $\xi \in \mathbb{R}^n$ with $\|\xi\| = 1$ the spectrum of $a_\pi(x, \eta, \xi) \subset \mathbf{C}_+ := \{z \in \mathbf{C} \mid Re\, z > 0\}$. We also assume that $\mathcal{B}$ satisfies the normal complementing condition (Lopatinskii-Shapiro condition) with respect to $\mathcal{A}$. This means that for each (x, ξ) in the tangent bundle of $\partial\Omega$ and each $\lambda \in \mathbf{C}_+$ with $(\xi, \lambda) \neq (0, 0)$, 0 is the only exponentially decaying solution on the half line for:

$$[\lambda + a_\pi(x, \xi + \nu(x) i\partial_t)]u = 0, \quad t > 0, \qquad b_\pi(x, \xi + \nu(x) i\partial_t)u(0) = 0.$$

It is not difficult to see that system (1)-(4) satisfies these restrictions. Consider the problem

$$\begin{gathered} \partial_t u + \mathcal{A}(u)u = f(\cdot, u) \quad \text{in } \Omega \times (0, \infty) \\ \mathcal{B}(u)u = g(\cdot, u) \quad \text{on } \partial\Omega \times (0, \infty) \\ u(\cdot, 0) = u_0 \quad \text{on } \Omega. \end{gathered} \tag{28}$$

where we assume that f and g are Lipschitz continuous. Let $W^s_q(\Omega)$ be the Sobolev-Slobodeckii space. We define

$$W^s_{q,\mathcal{B}} := \{(w_1, w_2, \dots, w_m) \mid w_i \in W^s_q(\Omega) \quad \text{and} \quad \mathcal{B}(w)w = g(\cdot, w)\, \forall i\}$$

We say that $u : [0, T] \to W^s_{q,\mathcal{B}}$ is a weak $W^s_{q,\mathcal{B}}$-solution of the above problem on $[0, T]$ if

$$u \in C([0, T], W^{s-2}_{q,\mathcal{B}}) \cap C((0, T), W^s_{q,\mathcal{B}} \cap C^1((0, T), W^{s-2}_{q,\mathcal{B}}),$$

and satisfies $u(0) = u_0$. We then have the following existence theorem.

Theorem 5.1 ([3]) *Suppose that $n/q < s < (1 + 1/q) \wedge (2 - n/q)$. Then the above boundary value problem has for each $u_0 \in W^s_{q,\mathcal{B}}(\Omega)$ a unique maximal weak $W^s_{q,\mathcal{B}}(\Omega)$-solution. If this solution remains bounded in $W^\rho_{\rho,\mathcal{B}}$ for some $\rho > 1$ then the solution exists on all of $[0, \infty)$. Moreover, if $g \equiv 0$ then the solution is in fact a classical solution. That is to say*

$$u \in C(\overline{\Omega} \times [0, T]) \cap C^{2,1}(\overline{\Omega} \times (0, T)),$$

where $u(0) = u_0$ and u satisfies the parabolic partial differential equation and boundary conditions pointwise.

The hypotheses we have imposed imply that problem (1)-(5) satisfies all the hypotheses of the above theorem. Moreover, since we reduced the problem to one with homogeneous boundary conditions we have the following result.

Theorem 5.2 *Problem (1)-(5) has a classical solution defined on a maximal interval $[0, T)$. If the initial conditions are nonnegative then all components of u and S will remain nonnegative for $0 < t < T$. If $T < \infty$ then for any $\rho > 1$, $\|(S, u)\|_{W^\rho_\rho(\Omega)}$ will attain arbitrarily large values as $t \uparrow T$.*

The positivity assertion can, for example, be deduced from the results in section 15 in [3]. In order to obtain a global existence result we use results from [10, chapter 4, section 9]. Because we will not need it in as much generality as is allowed in [10], we can simplify its statement considerably. Define $Q_T := \Omega \times [0, T)$. The space $W_p^{2,1}(Q_T)$ consists of all functions v such that $\partial_t v, \partial_i v, \partial_i \partial_j v \in L_p(Q_T)$ for all $1 \le i, j \le n$. The norm on this space is

$$\|v\|_{p,2,1} := \|v\|_p + \|\partial_t v\|_p + \sum_{i=1}^{n} \|\partial_i v\|_p + \sum_{i,j=1}^{n} \|\partial_i \partial_j v\|_p$$

Theorem 5.3 ([10, pp 341-351]) *Consider the equation*

$$\partial_t v - \Delta v + b_j(x)\partial_j v + a(x)v = f \in L_p(Q_T),$$

with boundary condition

$$\partial_\nu v + \sigma(x)v = 0$$

and with an initial condition $v(\cdot, 0) = v_0 \in W_p^2(\Omega)$ *that satisfies the boundary condition. Suppose that*

$$b_i \in L_{r_1}(Q_T), \quad a \in L_{r_2}(Q_T), \quad \sigma \in C^2(\overline{\Omega})$$

with $p \neq n+2$,$p \neq n/2+1$, $p \neq 3$, $r_1 = \max(p, n+2)$ *and* $r_2 = \max(p, n/2+1)$. *Then* $v \in W_p^{2,1}(Q_T)$.

Using this regularity result, it is possible to obtain the following result.

Theorem 5.4 *Suppose that* $E_q \le 0$ *and that there exists a continuous function* A *such that for all* $j = 0, 1, \dots, m$

$$F_j(x, t, S, u), \le A(S)[1 + |u|^Q], \tag{29}$$

with $q > (n+2)Q/2$ *and* $q > (n+1)(Q-1) \ge 0$. *Then problem (1)-(5) has a classical solution on* $[0, \infty)$. *This is also true in case* $E_q \ge 0$ *provided that for some* $0 < \delta < 1$, $\|\eta\|_{C^{1+\delta}(\Omega)}$ *and* $\|S_0\|_\infty$ *are sufficiently small. In both cases there are constants* C *and* μ *such that*

$$\|u(\cdot, t)\|_q \le C \exp(\mu t).$$

Proof. Again, for the sake of simplicity of notation we do the proof only for the case $m = 1$. Let T be the maximum value so that our problem has a solution on $[0, T)$. We will assume that $T < \infty$ and arrive at a contradiction. Note that s is a solution of a scalar parabolic equation with a source term in $L_{q/Q}(Q_T)$ and therefore, by theorem 5.3, s and S are members of $W_{q/Q}^{2,1}(Q_T)$. Define the function

$$\psi(p) := \frac{(n+1)p}{(n+1)Q - p}$$

In case the $q < (n+1)Q$, the Sobolev embedding theorem tells us that s, S, ∇s, and ∇S are in $L_{\psi(q)}(Q_T)$. If $q \geq (n+1)Q$ these functions are in $L_R(Q_T)$ for arbitrarily large R. It is easy to see that whenever $(n+1)Q > q > (n/2+1)Q$ then $\psi(q) > n+2$ hence $\Delta S \in L_{q/Q}(Q_T) \subset L_{n/2+1}(Q_T)$ and $|\nabla S| \in L_{\psi(q)}(Q_T) \subset L_{n+2}(Q_T)$ so that we may apply theorem 5.3 to the equation for u:

$$\partial_t u = d\Delta u - (\nabla B + \chi \nabla S) \cdot \nabla u - (\chi \Delta S + \chi' |\nabla S|^2 + \lambda) u + F(x, t, S, u).$$

Hence $u \in W^{2,1}_{q/Q}(Q_T)$ and consequently u and ∇u are members of $L_{\psi(q)}(Q_T)$ in the case $q < (n+1)Q$ or otherwise are members of $L_R(Q_T)$ for arbitrarily large R. It is easily seen that the function ψ has an unstable fixed point at $q^* := (n+1)(Q-1)$ and therefore, since $q > q^*$ we have $\psi(q) > q$. In the case $\psi(q) < (n+1)Q$, we repeat the above procedure with q being replaced by $\psi(q)$. Indeed we may keep iterating, thus bootstrapping until we have $S, u \in W^{2,1}_{P^*/Q}(Q_T)$ for some $P^* > Q(n+1)$. A downward adjustment of an iterate may be required along the way in order to avoid the value 3 which is disallowed by theorem 5.3. So eventually we have u, S, s, ∇u, ∇S, $\nabla s \in W^1_{P^*/Q}(Q_T) \subset L_\infty(Q_T)$. Since our system satisfies the conditions of [3, theorem 15.5] it follows that $T = \infty$.

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