

## EIGENVALUE PROBLEMS FOR THE P-LAPLACIAN WITH INDEFINITE WEIGHTS

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ABSTRACT. We consider the eigenvalue problem  $-\Delta_p u = \lambda V(x)|u|^{p-2}u$ ,  $u \in W_0^{1,p}(\Omega)$  where  $p > 1$ ,  $\Delta_p$  is the p-Laplacian operator,  $\lambda > 0$ ,  $\Omega$  is a bounded domain in  $\mathbb{R}^N$  and  $V$  is a given function in  $L^s(\Omega)$  ( $s$  depending on  $p$  and  $N$ ). The weight function  $V$  may change sign and has nontrivial positive part. We prove that the least positive eigenvalue is simple, isolated in the spectrum and it is the unique eigenvalue associated to a nonnegative eigenfunction. Furthermore, we prove the strict monotonicity of the least positive eigenvalue with respect to the domain and the weight.

### 1. INTRODUCTION

In this work we study the nonlinear eigenvalue problem

$$\begin{aligned} -\Delta_p u &= \lambda V(x)|u|^{p-2}u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where  $p > 1$ ,  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$  denotes the  $p$ -Laplacian,  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ ,  $V$  is a given function which may change sign and  $\lambda$  is the eigenvalue parameter. We assume that

$$\begin{aligned} V^+ &\not\equiv 0 \quad \text{and} \\ V &\in L^s(\Omega) \text{ for some } s > \frac{N}{p} \text{ if } 1 < p \leq N \text{ and } s = 1 \text{ if } p > N. \end{aligned} \tag{1.2}$$

As usual  $V^\pm(x) = \max\{\pm V(x), 0\}$ . We are interested in positive eigenvalues.

The goal of this paper is the study of the main properties (isolation, simplicity) of the *least positive eigenvalue*,

$$\lambda_1 \stackrel{\text{def}}{=} \inf \left\{ \int_{\Omega} |\nabla u|^p dx : u \in W_0^{1,p}(\Omega) \text{ and } \int_{\Omega} V |u|^p dx = 1 \right\}. \tag{1.3}$$

We prove that  $\lambda_1$  is associated to a  $C_{loc}^\alpha(\Omega)$  eigenfunction which is positive in  $\Omega$  and unique (up to a multiplicative constant). Moreover  $\lambda_1$  is the unique positive eigenvalue associated to a nonnegative eigenfunction.

These properties are well known in the case of bounded weights (see [3] for indefinite weights and [15] for the case  $V \equiv 1$ ). For non-negative weights satisfying

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(1.2) see [1, 10], and for indefinite weights with different integrability conditions see [2, 19].

Of course the main difficulty to prove the different properties of  $\lambda_1$  is the lack of regularity of the eigenfunctions. The results (as far as we know) concerning the regularity of weak solutions to degenerate elliptic equations are proved by [13, 14, 16]. These authors prove, for a class of degenerate quasilinear problems more general than the one considered here, that solutions of (1.1) are essentially bounded in  $\Omega$  and at least of class  $C_{loc}^\alpha(\Omega)$  for some  $0 < \alpha < 1$ . In the case of a bounded weight one can prove better results. In fact the results of [20],[21] and [9] imply that the solutions of problem (1.1) for  $V$  bounded are at least of class  $C_{loc}^{1,\alpha}$  (see Remark 2.3).

This lack of regularity can be a handicap if one wants to use for instance ‘‘Diaz-Saa’s inequality’’, which is a classical tool to prove the simplicity, or the ‘‘strong maximum principle’’ of Vazquez, a property which is used repeatedly in this context. We will show in this paper how to deal with this lack of regularity by using for instance ‘‘Picone’s identity’’ instead of Diaz-Saa’s inequality and ‘‘Harnack’s inequality’’ instead of Vazquez’s results.

This paper is organized as follows. In section 2 we recall some results about the existence of sequences of eigenvalues for problem (1.1). We also recall some regularity results that we will use later. In section 3 we give some basic properties of  $\lambda_1$  and we study the sign of the eigenfunctions. In section 4 we study simplicity, isolation and monotonicity properties of  $\lambda_1$ . We conclude this work in section 5 where we comment on some new results from [5, 11] and [4] on the second positive eigenvalue.

This work is mainly motivated by the study of asymmetric elliptic problems with weights done in [5]. Some of the results proved here were announced in that paper.

## 2. PRELIMINARIES

Throughout this paper  $\Omega$  will be a bounded domain of  $\mathbb{R}^N$  and we will always assume that condition (1.2) is satisfied.

$W_0^{1,p}(\Omega)$  will denote the usual Sobolev space with norm  $\|u\| = (\int_\Omega |\nabla u|^p dx)^{1/p}$ . We will write  $\|\cdot\|_p$  for the  $L^p$ -norm.  $\langle \cdot, \cdot \rangle$  will denote the duality product between  $W_0^{1,p}(\Omega)$  and its dual  $W^{-1,p'}(\Omega)$ .

We will write  $Y = L^{s'p}(\Omega)$  if  $1 < p \leq N$  and  $Y = C(\Omega)$  if  $p > N$ . The Lebesgue norm of  $Y$  (or the infinity norm in the case  $Y = C(\Omega)$ ) will be denoted by  $\|\cdot\|_Y$ . Notice that hypothesis (1.2) on  $s$  implies that the Sobolev imbedding  $i : W_0^{1,p}(\Omega) \hookrightarrow Y$  is compact.

We will also denote  $p' = \frac{p}{p-1}$  the Hölder conjugate exponent of  $p$  and  $p^*$  the critical exponent, that is  $p^* = \infty$  if  $p \geq N$  and  $p^* = \frac{Np}{N-p}$  if  $1 < p < N$ .

If  $A \subset \mathbb{R}^N$  is a measurable set,  $|A|$  denotes the Lebesgue measure in  $\mathbb{R}^N$ .

We recall that a value  $\lambda \in \mathbb{R}$  is an *eigenvalue* of problem (1.1) if and only if there exists  $u \in W_0^{1,p}(\Omega) \setminus \{0\}$  such that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi dx = \lambda \int_\Omega V |u|^{p-2} u \varphi dx \quad (2.1)$$

for all  $\varphi \in W_0^{1,p}(\Omega)$ .  $u$  is then called an *eigenfunction associated to  $\lambda$* . It is easy to see that the set of eigenvalues, called the *spectrum of (1.1)*, is a closed subset in  $\mathbb{R}$ .

Let us formulate variationally problem (1.1). For that purpose we introduce the  $C^1$  functionals  $\Phi$  and  $J : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\Phi(u) \stackrel{\text{def}}{=} \int_{\Omega} |\nabla u|^p dx \quad \text{and} \quad J(u) \stackrel{\text{def}}{=} \int_{\Omega} V |u|^p dx.$$

$J$  is well defined as we have, for all  $u \in W_0^{1,p}(\Omega)$ ,  $|J(u)| \leq \|V\|_s \|u\|_Y^p$ . Notice that  $J$  is indefinite if  $V$  changes sign.

It follows from the previous definitions that a real value  $\lambda$  is an eigenvalue of problem (1.1) if and only if there exists  $u \in W_0^{1,p}(\Omega) \setminus \{0\}$  such that  $\Phi'(u) = \lambda J'(u)$ . At this point let us introduce the set

$$\mathcal{M} \stackrel{\text{def}}{=} \{u \in W_0^{1,p}(\Omega) : J(u) = 1\}.$$

Condition  $V^+ \not\equiv 0$  from (1.2) implies that  $\mathcal{M} \neq \emptyset$ . Moreover the set  $\mathcal{M}$  is a manifold in  $W_0^{1,p}(\Omega)$  of class  $C^1$ . For any  $u \in \mathcal{M}$  the tangent space of  $\mathcal{M}$  at  $u$ ,  $T_u\mathcal{M}$ , is the set  $T_u\mathcal{M} = \{w \in W_0^{1,p}(\Omega) : \langle J'(u), w \rangle = 0\}$ . Let us denote by  $\tilde{\Phi}$  the restriction of  $\Phi$  to  $\mathcal{M}$ . We recall that a value  $c$  is a *critical value of  $\tilde{\Phi}$*  if  $\Phi'(u)|_{T_u\mathcal{M}} \equiv 0$  and  $\tilde{\Phi}(u) = c$  for some  $u \in \mathcal{M}$ .

It follows from standard arguments that positive eigenvalues of (1.1) correspond to positive critical values of  $\tilde{\Phi}$ . A first sequence of positive critical values of  $\tilde{\Phi}$  comes from the Ljusternik-Schnirelman critical point theory on  $C^1$  manifolds proved by [18]. That is, if  $\gamma(A)$  denotes the Krasnoselski's genus on  $W_0^{1,p}(\Omega)$  and for any  $k \in \mathbb{N}_*$  we set  $\Gamma_k \stackrel{\text{def}}{=} \{A \subset \mathcal{M} : A \text{ is compact, symmetric and } \gamma(A) \geq k\}$ , then the value

$$\lambda_k \stackrel{\text{def}}{=} \inf_{A \in \Gamma_k} \max_{u \in A} \Phi(u) \tag{2.2}$$

is an eigenvalue of (1.1). Moreover  $\lim_{k \rightarrow +\infty} \lambda_k = +\infty$ .

**Remark 2.1.** One can also define another sequence of critical values minimaxing  $\tilde{\Phi}$  along a smaller family of symmetric subsets of  $\mathcal{M}$ . The following result can be proved using the minimax principle of [6]. Let us denote by  $S^k$  the unit sphere of  $\mathbb{R}^{k+1}$  and

$$\mathcal{O}(S^k, \mathcal{M}) \stackrel{\text{def}}{=} \{h \in C(S^k, \mathcal{M}) : h \text{ is odd}\}.$$

Then for any  $k \in \mathbb{N}_*$  the value

$$\mu_k \stackrel{\text{def}}{=} \inf_{h \in \mathcal{O}(S^{k-1}, \mathcal{M})} \max_{z \in S^{k-1}} \Phi(h(z)) \tag{2.3}$$

is an eigenvalue of (1.1). Moreover  $\lambda_k \leq \mu_k$ .

This new sequence of eigenvalues was first introduced in the case of  $V \equiv 1$  by [11] and was used there to establish resonance and nonresonance results associated to the  $p$ -laplacian operator. More recently, this sequence appears to be of some interest in the study of the Fučík spectrum of the  $p$ -laplacian made by the author in [7].

**Remark 2.2.** It is a trivial fact that

$$\lambda_1 = \mu_1 = \inf \left\{ \int_{\Omega} |\nabla u|^p dx : u \in W_0^{1,p}(\Omega) \text{ and } \int_{\Omega} V |u|^p dx = 1 \right\}.$$

We will see in section 5 that also  $\lambda_2 = \mu_2$ . Whether or not  $\lambda_k = \mu_k$  for other  $k$ 's is still an open question when  $p \neq 2$ . The proof that  $\lambda_k = \mu_k$  for all  $k \geq 1$  in the case  $p = 2$  is quite simple and it is left to the reader. When  $N = 1$ ,  $V \equiv 1$  and

$p > 1$  it is proved for instance in [7] that  $\lambda_k = \mu_k$  for all  $k \geq 1$  but this last equality remains an open question when  $N > 1$ .

We conclude this section recalling some results about the regularity and boundedness of the eigenfunctions of (1.1). The first part of the next proposition is proved in [13, Propositions 1.2 and 1.3] (see also in [14, Théorèmes 7.1-7.2, pg.262]) The second part can be found in [16, Theorem 8].

**Proposition 2.1.** [13, 16] *Let  $u \in W_0^{1,p}(\Omega) \setminus \{0\}$  be an eigenfunction associated to  $\lambda$ . Then (i)  $u \in L^\infty(\Omega)$  and (ii)  $u$  is locally Hölder continuous, that is, there exists  $\alpha = \alpha(p, N, \|\lambda V\|_s) \in ]0, 1[$  s.t. for any subdomain  $\Omega' \subset \Omega$  there exist  $C = C(p, N, \|\lambda V\|_s, \text{dist}(\Omega', \partial\Omega))$  such that*

$$|u(x) - u(y)| \leq C \|u\|_\infty |x - y|^\alpha, \quad \forall x, y \in \Omega'.$$

**Remark 2.3.** Under the hypothesis (1.2) on  $V$  we can not assure that the solutions of (1.1) are of class  $C_{loc}^{1,\alpha}$ . The  $C_{loc}^{1,\alpha}$  regularity proved by [20, 21, 9] could be applied here provided the weight  $V$  is either bounded or belongs to some  $L^r(\Omega)$  with  $r > Np'$ .

### 3. SIGN PROPERTIES OF THE EIGENVALUES

**Proposition 3.1.** *The infimum  $\lambda_1$  in (1.3) is achieved at some  $u \in \mathcal{M}$ ,  $\lambda_1 > 0$  and  $\lambda_1$  is the least positive eigenvalue of problem (1.1). Moreover  $\lambda_1 = \Phi(u)$  for some  $u \in \mathcal{M}$  if and only if  $u$  is an eigenfunction associated to  $\lambda_1$ .*

*Proof.* The proof is an straight application of Theorem 1.2 of [17] and the Lagrange's multiplier rule.  $\square$

The following "strong maximum principle" holds :

**Proposition 3.2.** *If  $u \in W_0^{1,p}(\Omega)$  is a non-negative weak solution of (1.1) then either  $u \equiv 0$  or  $u(x) > 0$  for all  $x \in \Omega$ .*

*Proof.* The result is a direct consequence of the following Harnack's inequality for nonnegative solutions of (1.1). We refer here to [16, Theorems 5,6 and 9, pg.264-270].

"Let  $u \in W^{1,p}(\Omega)$  be a non-negative weak solution of (1.1) and assume that  $B(x_0, 3r) \subset \Omega$  for some  $r > 0$  and  $x_0 \in \Omega$ . Then for some  $C = C(p, N, r, \lambda V, \Omega)$ ,

$$\max_{\overline{B}(x_0, r)} u \leq C \min_{\overline{B}(x_0, r)} u."$$

$\square$

We also have the following result :

**Proposition 3.3.** *The eigenfunctions associated to  $\lambda_1$  are either positive or negative in  $\Omega$ .*

*Proof.* Let  $u \in \mathcal{M}$  be an eigenfunction associated to  $\lambda_1$ . Then  $u$  achieves the infimum in (1.3). Since  $\|\nabla|u|\|_p = \|\nabla u\|_p$  and  $|u| \in \mathcal{M}$  it follows that  $|u|$  achieves also the infimum in (1.3) and therefore, from Proposition 3.1,  $|u|$  is an eigenfunction for  $\lambda_1$ . By Proposition 3.2 we conclude that  $|u(x)| > 0 \quad \forall x \in \Omega$  and consequently  $u$  is either positive or negative in  $\Omega$ .  $\square$

In what follows we will use the so-called "Picone's identity" proved in [2]. We recall it here for completeness.

**Theorem 3.1.** [2] *Let  $v > 0, u \geq 0$  be two continuous functions in  $\Omega$  differentiable a.e. Denote*

$$L(u, v) = |\nabla u|^p + (p - 1) \frac{u^p}{v^p} |\nabla v|^p - p \frac{u^{p-1}}{v^{p-1}} |\nabla v|^{p-2} \nabla v \nabla u,$$

$$R(u, v) = |\nabla u|^p - |\nabla v|^{p-2} \nabla \left( \frac{u^p}{v^{p-1}} \right) \nabla v.$$

*Then (i)  $L(u, v) = R(u, v)$ , (ii)  $L(u, v) \geq 0$  a.e. and (iii)  $L(u, v) = 0$  a.e. in  $\Omega$  if and only if  $u = kv$  for some  $k \in \mathbb{R}$ .*

In the next theorem we give an estimate of the measure of the nodal domains of an eigenfunction  $u$ . We recall that a *nodal domain* of  $u$  is a connected component of  $\Omega \setminus \{x \in \Omega : u(x) = 0\}$ . The same result for positive weights can be found in [1]. Our exponent  $\gamma$  is slightly different.

**Theorem 3.2.** *Any eigenfunction  $v$  associated to a positive eigenvalue  $0 < \lambda \neq \lambda_1$  changes sign. Moreover if  $\mathcal{N}$  is a nodal domain of  $v$  then*

$$|\mathcal{N}| \geq (C\lambda \|V\|_s)^{-\gamma} \tag{3.1}$$

where  $\gamma = \frac{sN}{sp-N}$  and  $C$  is some constant depending only on  $N$  and  $p$  if  $p \neq N$  and on  $N$  and  $s'$  if  $p = N$ .

*Proof.* Assume by contradiction that  $v \geq 0$ , the case  $v \leq 0$  being completely analogous. By Proposition 3.2 it follows that  $v(x) > 0$  for all  $x \in \Omega$ . Let  $\varphi > 0$  be an eigenfunction associated to  $\lambda_1$ . For any  $\epsilon > 0$  we apply Picone's identity to the pair  $\varphi, v + \epsilon$ . We have

$$\begin{aligned} 0 &\leq \int_{\Omega} L(\varphi, v + \epsilon) dx = \int_{\Omega} R(\varphi, v + \epsilon) dx = \\ &\lambda_1 \int_{\Omega} V \varphi^p dx - \int_{\Omega} |\nabla v|^{p-2} \nabla \left( \frac{\varphi^p}{(v+\epsilon)^{p-1}} \right) \nabla v dx. \end{aligned} \tag{3.2}$$

Notice that  $\frac{\varphi^p}{(v+\epsilon)^{p-1}}$  belongs to  $W_0^{1,p}(\Omega)$  and then it is admissible in the weak formulation of  $-\Delta_p v = \lambda V|v|^{p-2}v$ . Then it follows from (3.2) that

$$0 \leq \int_{\Omega} V \varphi^p \left( \lambda_1 - \lambda \frac{v^{p-1}}{(v+\epsilon)^{p-1}} \right) dx.$$

Letting  $\epsilon \rightarrow 0$  it comes that  $0 \leq \int_{\Omega} V \varphi^p (\lambda_1 - \lambda) dx$  which is impossible because  $\lambda > \lambda_1$  and  $\int_{\Omega} V \varphi^p dx > 0$ . Hence we have proved that  $v$  must change sign.

Next we prove estimate (3.1). Assume that  $v > 0$  in  $\mathcal{N}$ , the case  $v < 0$  being completely analogous. We observe that because  $v \in W_0^{1,p}(\Omega) \cap C(\Omega)$  then  $v|_{\mathcal{N}} \in W_0^{1,p}(\mathcal{N})$ . Hence the function  $w$  defined as  $w(x) = v(x)$  if  $x \in \mathcal{N}$  and  $w(x) = 0$  if  $x \in \Omega \setminus \mathcal{N}$  belongs to  $W_0^{1,p}(\Omega)$ .

Let us start with the case  $1 < p < N$ . Using  $w$  as a test function in the weak equation satisfied by  $v$  we find

$$\int_{\mathcal{N}} |\nabla v|^p dx = \lambda \int_{\mathcal{N}} V |v|^p dx \leq \lambda \|V\|_s \|v\|_{p^*, \mathcal{N}}^p |\mathcal{N}|^{\frac{p^* - s'p}{s'p^*}}$$

by Hölder inequality. On the other hand using Sobolev imbeddings we have that  $\int_{\mathcal{N}} |\nabla v|^p dx \geq C \|v\|_{p^*, \mathcal{N}}^p$  for some constant  $C = C(N, p)$ . Hence

$$C \leq \lambda \|V\|_s |\mathcal{N}|^{\frac{p^* - s'p}{s'p^*}}$$

and the proposition follows.

In the case  $p = N$  we proceed similarly using Sobolev's inclusion  $W_0^{1,N}(\mathcal{N}) \subset L^{Ns'}(\mathcal{N})$ , the estimate (7.38) of [12] and then apply Hölder inequality. We find

$$C\|v\|_{N^{s'},\mathcal{N}}^N |\mathcal{N}|^{-1/s'} \leq \int_{\mathcal{N}} |\nabla v|^N dx \leq \lambda \|V\|_s \|v\|_{N^{s'},\mathcal{N}}^N$$

for some  $C = C(N, s')$  and then inequality (3.1) follows.

In the case  $p > N$  we have on the one hand

$$\int_{\mathcal{N}} |\nabla v|^p dx \leq \lambda \|V\|_1 \|v\|_{\infty,\mathcal{N}}^p,$$

and on the other hand, from Morrey's lemma,

$$C\|v\|_{\infty,\mathcal{N}} \leq |\mathcal{N}|^{-1/p+1/N} \|\nabla v\|_{p,\mathcal{N}}$$

for some  $C = C(N, p)$ . Then inequality (3.1) holds.  $\square$

**Corollary 3.1.** *Each eigenfunction has a finite number of nodal domains.*

*Proof.* Let  $\mathcal{N}_j$  be a nodal domain of an eigenfunction associated to some positive eigenvalue  $\lambda$ . It follows from (3.1) that

$$|\Omega| \geq \sum_j |\mathcal{N}_j| \geq (C\lambda \|V\|_s)^{-\gamma} \sum_j 1$$

and the claim follows.  $\square$

#### 4. ON THE FIRST EIGENVALUE

We have proved in the previous section that the eigenfunctions associated to  $\lambda_1$  have definite sign in  $\Omega$ . We are now going to prove that this property implies, through Picone's identity, that  $\lambda_1$  is simple. We will also prove that  $\lambda_1$  is isolated in the spectrum of (1.1) as a consequence of Theorem 3.2.

Finally we give a result on the strict monotonicity of  $\lambda_1$  with respect to both the domain and the weight.

**Proposition 4.1.**  *$\lambda_1$  is simple in the sense that the eigenfunctions associated to it are merely a constant multiple of each other.*

*Proof.* We proceed as in the first part of the proof of Theorem 3.2. Let  $u, v$  be two eigenfunctions associated to  $\lambda_1$ . We can assume without restriction that  $u$  and  $v$  are positive in  $\Omega$ . Let  $\epsilon > 0$ . From Picone's identity we have

$$0 \leq \int_{\Omega} L(u, v + \epsilon) dx = \int_{\Omega} R(u, v + \epsilon) dx = \lambda_1 \int_{\Omega} V u^p dx - \int_{\Omega} |\nabla v|^{p-2} \nabla \left( \frac{u^p}{(v+\epsilon)^{p-1}} \right) \nabla v dx.$$

The function  $\frac{u^p}{(v+\epsilon)^{p-1}}$  belongs to  $W_0^{1,p}(\Omega)$  and then it is admissible for the weak formulation of  $-\Delta_p v = \lambda_1 V |v|^{p-2} v$ . It follows then from the previous equation that

$$0 \leq \int_{\Omega} L(u, v + \epsilon) dx = \lambda_1 \int_{\Omega} V u^p \left( 1 - \frac{v^{p-1}}{(v+\epsilon)^{p-1}} \right) dx.$$

Letting  $\epsilon \rightarrow 0$  it follows that  $L(u, v) = 0$ . Then, by Theorem 3.1, there exists  $k \in \mathbb{R}$  such that  $u = kv$ .  $\square$

**Proposition 4.2.**  *$\lambda_1$  is isolated, that is, there exists  $\delta > 0$  such that in the interval  $(\lambda_1, \lambda_1 + \delta)$  there are no other eigenvalues of (1.1).*

*Proof.* The result follows easily from the estimate (3.1). Assume by contradiction that there exists a sequence of eigenvalues of (1.1)  $\lambda_n$  with  $0 < \lambda_n \searrow \lambda_1$ . Let  $u_n$  be an eigenfunction associated to  $\lambda_n$ . Since  $0 < \int_{\Omega} |\nabla u_n|^p dx = \lambda_n \int_{\Omega} V |u_n|^p dx$  we can define

$$v_n := \frac{u_n}{\left(\int_{\Omega} V |u_n|^p dx\right)^{1/p}}.$$

$v_n$  is bounded in  $W_0^{1,p}(\Omega)$  so there exist a subsequence (still denoted  $v_n$ ) and  $v \in W_0^{1,p}(\Omega)$  such that  $v_n \rightarrow v$  in  $Y$  and weakly in  $W_0^{1,p}(\Omega)$ . Moreover  $\int_{\Omega} V |v|^p dx = 1$ . On the other hand

$$\int_{\Omega} |\nabla v|^p dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} |\nabla v_n|^p dx = \lambda_1$$

and then  $\int_{\Omega} |\nabla v|^p dx = \lambda_1$  by (1.3). Using Proposition 3.1 we conclude that  $v$  is an eigenfunction associated to  $\lambda_1$ . It follows then from Proposition 3.3 that either  $v > 0$  or  $v < 0$ . In the case  $v > 0$  (the other case is analogous) we conclude from the convergence in measure of the sequence  $v_n$  towards  $v$  that

$$|\Omega_n^-| \rightarrow 0 \tag{4.1}$$

where  $\Omega_n^-$  denotes the negative set of  $u_n$ . But (4.1) clearly contradicts estimate (3.1).  $\square$

In the sequel we will denote the least positive eigenvalue of (1.1) by  $\lambda_1(V)$  or  $\lambda_1(\Omega)$  when comparing  $\lambda_1$  for different weights or domains.

We will always assume that condition (1.2) is satisfied for the weights appearing in the claims.

**Proposition 4.3.** *Let  $V_1, V_2$  be two weights and assume that  $V_1 \leq V_2$  a.e. and  $|\{x \in \Omega : V_1(x) < V_2(x)\}| \neq 0$ . Then  $\lambda_1(V_2) < \lambda_1(V_1)$ .*

*Proof.* Let  $u > 0$  be an eigenfunction associated to  $\lambda_1(V_1)$ . Since

$$0 < \lambda_1(V_1)^{-1} \int_{\Omega} |\nabla u|^p dx = \int_{\Omega} V_1 u^p dx \leq \int_{\Omega} V_2 u^p dx,$$

we can use  $u/(\int_{\Omega} V_2 u^p dx)^{1/p}$  as an admissible function in the infimum of (1.3) for  $\lambda_1(V_2)$ . We have

$$\lambda_1(V_2) \leq \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} V_2 u^p dx} \leq \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} V_1 u^p dx} = \lambda_1(V_1).$$

Thus  $\lambda_1(V_2) \leq \lambda_1(V_1)$ . The equality holds if and only if  $\int_{\Omega} V_1 u^p dx = \int_{\Omega} V_2 u^p dx$ . This last identity implies that  $V_1 \equiv V_2$  because  $u > 0$  in  $\Omega$ , contradicting our hypothesis.  $\square$

**Proposition 4.4.** *Let  $\Omega_1$  be a proper open subset of a domain  $\Omega_2 \subset \mathbb{R}^N$ . Then  $\lambda_1(\Omega_2) < \lambda_1(\Omega_1)$ .*

*Proof.* Let  $u \in W_0^{1,p}(\Omega_1)$  be a positive eigenfunction associated to  $\lambda_1(\Omega_1)$  and put  $\tilde{u}$  the function obtained by extending  $u$  by zero in  $\Omega_2 \setminus \Omega_1$ . Then  $\tilde{u} \in W_0^{1,p}(\Omega_2)$  and  $\int_{\Omega_2} V \tilde{u}^p dx = \int_{\Omega_1} V u^p dx > 0$ . Using  $\tilde{u}/(\int_{\Omega_2} V \tilde{u}^p dx)^{1/p}$  as an admissible function for  $\lambda_1(\Omega_2)$  we get

$$\lambda_1(\Omega_2) \leq \frac{\int_{\Omega_2} |\nabla \tilde{u}|^p dx}{\int_{\Omega_2} V \tilde{u}^p dx} = \frac{\int_{\Omega_1} |\nabla u|^p dx}{\int_{\Omega_1} V u^p dx} = \lambda_1(\Omega_1)$$

The equality holds only if  $\tilde{u}$  is an eigenfunction associated to  $\lambda_1(\Omega_2)$  but this is impossible because  $|\tilde{u} = 0| > 0$  in contradiction with Proposition 3.3.  $\square$

**Remark 4.1.** In [19] the authors proved the existence of a least positive eigenvalue for indefinite weights satisfying an integrability condition which is less restrictive than condition (1.2). Precisely they consider the case  $1 < p < N$  and an indefinite  $L^1_{loc}$ -weight having a positive part  $V^+ = V_1 + V_2$  with  $V_1 \in L^{\frac{N}{p}}(\Omega)$  and  $\lim_{x \rightarrow y} |x - y|^p V_2(x) = 0 \quad \forall y \in \overline{\Omega}$ . Thus, in the particular case  $V_2 \equiv 0$ , their hypothesis on  $V = V_1$  is weaker than ours. Concerning the properties of the least positive eigenvalue, the authors only proved that the associated eigenfunctions have definite sign.

## 5. FINAL COMMENTS AND REMARKS

Since  $\lambda_1$  is isolated in the spectrum and there exist eigenvalues different from  $\lambda_1$ , it makes sense to define the *second eigenvalue* of (1.1) as

$$\underline{\lambda}_2 \stackrel{\text{def}}{=} \min\{\lambda \in \mathbb{R} : \lambda \text{ eigenvalue and } \lambda > \lambda_1\}.$$

There exist several variational characterizations of  $\underline{\lambda}_2$  through mimimax formulae. For instance in [4] it is proved that  $\underline{\lambda}_2 = \lambda_2$  when  $V \in L^\infty(\Omega)$  and in [11] that  $\underline{\lambda}_2 = \mu_2$  when  $V \equiv 1$ . A further variational characterization has been given by [8] in the case  $V \equiv 1$ . This last characterization has been generalized recently by [5] to weights as those considered here. The following result was obtained as a consequence of the construction of the first curve of the Fučík spectrum in [5].

**Theorem 5.1.** [5] *Assume that  $V$  satisfies (1.2). Then*

$$\underline{\lambda}_2 = \inf_{h \in \mathcal{F}} \max_{u \in h([-1,1])} \int_{\Omega} |\nabla u|^p dx$$

where  $\mathcal{F} \stackrel{\text{def}}{=} \{\gamma \in C([-1, +1], \mathcal{M}) : \gamma(\pm 1) = \pm \varphi_1\}$  and  $\varphi_1 \in \mathcal{M}$  is the positive eigenfunction associated to  $\lambda_1$ .

**Remark 5.1.** Notice that  $\mathcal{F} \subset \Delta_2 \subset \Gamma_2$ . The variational characterization of Theorem 5.1 is slightly better than the one of [4] and [11] because it suffices to minimize along a smaller family of subsets of  $\mathcal{M}$  to get the same value.

A straight consequence of this result is the following :

**Corollary 5.1.**  $\underline{\lambda}_2 = \lambda_2 = \mu_2 = \inf_{h \in \mathcal{F}} \max_{u \in h([-1,1])} \int_{\Omega} |\nabla u|^p dx.$

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