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ON A GENERALIZED REFLECTION LAW FOR FUNCTIONS SATISFYING THE HELMHOLTZ EQUATION

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ABSTRACT. We investigate a generalized point to point reflection law for the solutions of the Helmholtz equation in two independent variables, obtaining results that include some previously known results of Khavinson and Shapiro as special cases. As a consequence, we obtain partial negative answers to the "point to compact set reflection" conjecture suggested by Garabedian and others.

1. INTRODUCTION

According to a result of H. Lewi [7], if u is a function of two variables and satisfies a partial differential equation with the Laplacian in the principal part in a domain D adjacent to the real axis $\mathbb{R}$ and $u_{|_{\mathbb{R}}} = 0$, then u extends to a mirror image D' of D with respect to $\mathbb{R}$. However, with regard to point to point "reflection laws", the situation for operators slightly different from the Laplacian (or the wave operator) is drastically different. The following theorem is due to Khavinson and Shapiro, [6] (see also [5], and compare with Study's interpretation of the Schwarz reflection principle).

Theorem 1.1. *Let $\gamma = \{(s,t) \mid \ t = g(s)\}$, with $g'(s) \neq 0$, be a non-singular real analytic curve in $\mathbb{R}^2$. If for two points P and Q sufficiently close to γ (but off γ) there exist a constant $c = c(P,Q)$ such that the "Schwarz reflection law"*

$$u(P) + c\,u(Q) = 0 \tag{1.1}$$

holds $\forall u, u_{|_\gamma} = 0$ and satisfying the "Helmholtz equation"

$$\frac{\partial^2 u}{\partial s \partial t} + \lambda^2 u = 0, \tag{1.2}$$

where $\lambda > 0$, then $c = 1$, P and Q must be symmetric with respect to γ and γ must be a straight line.

By using Study's change of variables $z = X + iY$, $w = X - iY$ that reduces the "real" Helmholtz operator $\triangle + \lambda^2$ to the complex hyperbolic operator $4\frac{\partial^2}{\partial z \partial w} + \lambda^2$, a similar conclusion as in Theorem 1.1 holds $\forall u, u|_\gamma = 0$ and satisfying $\triangle u + \lambda^2 u = 0$.

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Suppose Γ is a non-singular real analytic hypersurface defined in some domain $\Omega \subseteq \mathbb{R}^n$, and L is a partial differential operator in $\mathbb{R}^n$. Theorem 1.1 above can be considered as a partial answer to the first of the following two general problems:

(1) For which pair of points $P \neq Q$ in $\Omega \setminus \Gamma$ is it true that there is a constant $c = c(P, Q)$ such that

$$u(P) + cu(Q) = 0 \tag{1.3}$$

for all solutions of $Lu = 0$ in Ω, vanishing on Γ ? We say the pair P, Q has the *(point to point) reflection property* (with respect to Ω, Γ and L) if there is such a constant c.

(2) (In general when the point to point reflection property fails): Does a "point to compact set reflection" hold, i.e., given a point P_0, is there a compact set $K = K(P_0, \Gamma, L) \subseteq \Omega$ (on the "other side " of Γ) and a measure (or distribution) $T = T(P_0, \Gamma, L)$ supported on K such that

$$u(P_0) = \langle T, u\rangle \text{ for all solutions of } Lu = 0 \text{ vanishing on } \Gamma \text{ ?}$$

Problems of type (1) have been studied by several researchers ([7], [3], [6], [2], [1]; see also references there). This problem was the theme of the paper [2] in the special case $L = \Delta$ (the Laplace operator) and is completely settled there for any dimension $n \geq 3$. There, it was shown that such pairs of points are very rare in $\mathbb{R}^n$ when n is even (and > 3), and that the reflection property never holds when n is odd (and ≥ 3), unless Γ is a sphere or a hyperplane, in contrast to the case $n = 2$ and the classical Schwarz reflection principle. When L is the Helmholtz operator and $n = 2$, no pair of points have the reflection property unless Γ is a line (this is Theorem 1.1 above, which is an implicit result in [6]). The case of the real Helmholtz operator for $n = 3$ was studied in [1], a striking result there (among others) being, in contrast to the $n = 2$ case, there are hypersurfaces Γ other than hyperplanes and pairs of points where the reflection property holds.

So far, no result related to the "point to compact set reflection" problem was published, although it was conjectured by Garabedian, and later by the authors of the papers [6] and [2], that it is likely to get positive results in the instances of failure of the point to point reflection property. An explicit calculation supporting this conjecture for the (real) Helmholtz operator in two dimensions has been done in [9] (or see their rather widely available book, [10]). More precisely, if Ω is a domain in $\mathbb{R}^2$ divided by a non-singular real analytic curve γ into two parts, a formula which gives the value at a point $P_0 \in \Omega \setminus \gamma$ of a function u satisfying the (real) Helmholtz equation near γ and vanishing on γ is given in terms of an integral involving u and its first order derivatives, evaluated along a path joining γ to the point symmetric to P_0 with respect to γ.

Our main result in this paper is Theorem 3.1, which says that under the same hypothesis as in Theorem 1.1, if the "Schwarz reflection law" given by (1.1) is replaced by the weaker hypothesis,

$$u(P) = \sum_{|\alpha| \leq N} c_\alpha \, D^\alpha u(Q) \ , \tag{1.4}$$

where N, c_α ($|\alpha| \leq N$) are constants depending only on P and Q, then the conclusion of the theorem remains the same. In other words, this means that the reflection property in the case of the two dimensional Helmholtz operator, which was known to fail for the "Schwarz reflection law" given by (1.1) (when γ is not a line), also fails even when this law is replaced by the weaker hypothesis given by (1.4), which we may refer to as the "generalized (Schwarz) reflection law". This entails similar failures in the case of the two dimensional "real" Helmholtz operator (when Γ is not a line) and in the case of the three dimensional Laplace operator when Γ is a non-planar cylinder, see Corollaries 3.3 (i), 3.5 (i). All these results generalize previously known results in [6]. As a consequence, related partial negative answers to the 'point to compact set reflection' conjecture mentioned above are obtained, i.e., in the case of the two dimensional (real) Helmholtz operator (when γ is not a line) and in the case of the three dimensional Laplace operator (when Γ is a non-planar cylinder), there is no point to point reflection in the sense of problem (2) mentioned above, see Corollaries 3.2, 3.3 (ii) and 3.5 (ii).

2. Preliminaries

If

$$L = \frac{\partial^2}{\partial s \partial t} + a(s,t)\frac{\partial}{\partial s} + b(s,t)\frac{\partial}{\partial t} + c(s,t)$$

is a hyperbolic differential operator, where a, b and c are entire functions of two variables, its adjoint is defined by

$$L^\star u = \frac{\partial^2 u}{\partial s \partial t} - \frac{\partial}{\partial s}(au) - \frac{\partial}{\partial t}(bu) + cu.$$

The Riemann function $R_L := R(s,t;x,y)$ at a point (x,y) for the operator L is defined as the solution of the following Cauchy-Goursat problem

$$\begin{gathered} L^\star R = 0 \text{ near } (x,y)\,, \\ R(x,t;x,y) = \exp \int_y^t a(x,\tau)\, d\tau\,, \\ R(s,y;x,y) = \exp \int_x^s b(\tau,y)\, d\tau\,. \end{gathered}$$

If we define $r(s) = R(s,y;x,y)$ and $s(t) = R(x,t;x,y)$, it is easy to see that $r_s - br = 0$ on $\{t = y\}$, $s_t - as = 0$ on $\{s = x\}$ and $R(x,y;x;y) = 1$. Moreover, it is known that $R(s,t;x,y)$ is an entire function of all four variables and if L is the Helmholtz operator, then

$$R(s,t;x,y) = J_0(2\lambda\sqrt{(s-x)(t-y)}\,)\,,$$

where J_0 is the zero Bessel function. For these and other properties of the Riemann function, we refer to [4]. Let γ be as in Theorem 1.1, $P = P(x,y)$ be a point sufficiently close to γ. Let

$$\begin{gathered} A_P = (g^{-1}(y), y),\ B_P = (x, g(x)) \\ U = aRu + \tfrac{1}{2}Ru_t - \tfrac{1}{2}R_t u \\ V = bRu + \tfrac{1}{2}Ru_s - \tfrac{1}{2}R_s u\,. \end{gathered}$$

By a straightforward calculation, we have

$$\begin{aligned} 0 &= R(Lu) - u(L^\star R) \\ &= U_s + V_t\,. \end{aligned}$$

If $\mathbf{G}$ denotes the region bounded by segments PA, PB and the arc AB, applying Green's theorem we obtain

$$\oint_{\partial \mathbf{G}} U dt - V ds = 0 \,.$$

Simplifying this using the properties of the Riemann function and using the fact that $u_{|_\gamma} = 0$, we obtain the following formula of Riemann:

$$u(P) = \frac{1}{2}\int_{A_P}^{B_P} R(\frac{\partial u}{\partial s}ds - \frac{\partial u}{\partial t}dt). \tag{2.1}$$

3. Main results

Theorem 3.1. *Suppose*

$$u(P) = \sum_{|\alpha|\le N} c_\alpha \, D^\alpha u(Q) \tag{3.1}$$

$\forall u, u_{|_\gamma} = 0$ *and satisfying the Helmholtz equation* (1.2)*, where* P *and* Q $(P \neq Q)$ *are points in* $\mathbb{R}^2$ *sufficiently close to* γ *(but off* γ*),* $c_\alpha = c_\alpha(P, Q)$ *and* $N = N(P, Q)$ *is an integer. Then,* $c_\alpha = 0 \ \forall \alpha \neq 0$*,* $c_0 = -1$*,* P *and* Q *must be symmetric with respect to* γ *and* γ *must be a straight line.*

Proof. Since $u_{xy} = -\lambda^2 u$, there is no loss of generality in assuming that there are no mixed derivatives involved in the hypothesis (3.1). So let

$$u(P) = \sum_{n=0}^{N} c_n (D_x^n u)(Q) + \sum_{n=1}^{N} d_n (D_y^n u)(Q)\,, \tag{3.2}$$

where c_n, d_n $(0 \le n \le N)$ are constants that depend only on P and Q. If P and Q are sufficiently close to γ, all the solutions of the Cauchy problem

$$\begin{gathered} \frac{\partial^2 u}{\partial s \partial t} + \lambda^2 u = 0\,, \\ \partial^\alpha u_{|_\gamma} = \partial^\alpha ((t - g(s))p(s,t))_{|_\gamma}, \ |\alpha| \le 1\,, \end{gathered} \tag{3.3}$$

with $p(s, t)$ being a polynomial, are real analytic in a fixed neighbourhood containing P and Q. For u satisfying the Cauchy data in (3.3) we have,

$$(\frac{\partial u}{\partial s}ds - \frac{\partial u}{\partial t}dt)_{|_\gamma} = -2pg'(s)_{|_\gamma} \,.$$

Using this, (2.1) reduces to

$$u(P) = -\int_{A_P}^{B_P} R(s,t;P)g'pds\,, \tag{3.4}$$

where $p(s, t)$ is a polynomial. Differentiating (3.4) n-times with respect to x, we obtain the following expression for the *n-th* derivative of u at the point P:

$$\begin{aligned} (D_x^n u)(P) = &-\int_{A_P}^{B_P} (D_x^n R)(s,t;P)g'pds \\ &- \sum_{k=0}^{n-1} D_x^k((D_x^{n-1-k}R)(B_P;P)g'(x)p(B_P)). \end{aligned} \tag{3.5}$$

Denote the sum in the right hand side of (3.5) by v_n $(n = 1, 2, \ldots, N)$ and note that we can write,

$$\sum_{n=1}^{N} c_n v_n = \sum_{|\alpha| \le N-1} M_\alpha \, (D^\alpha p)(B_P) \,, \tag{3.6}$$

with

$$M_{(n-1,0)} = c_n R(B_P; P) g'(x) + \sum_{j=n+1}^{N} c_j K_{n-1}^j, \;\; 1 \le n \le N \,, \tag{3.7}$$

where $M_{(n-1,0)}$ denotes the coefficient M_α that appears in the right hand side of (3.6) for $\alpha = (n-1, 0)$ $(1 \le n \le N)$ and K_{n-1}^j, whose exact values we need not compute, depend on $(D_x R)(B_P; P)$, $g'(x)$ and their derivatives. Similarly, we have

$$\begin{aligned}(D_y^n u)(P) = &-\int_{A_P}^{B_P} (D_y^n R)(s,t;P) g' p ds \\ &+ \sum_{k=0}^{n-1} D_y^k ((D_y^{n-1-k} R)(A_P; P) p(A_P)).\end{aligned} \tag{3.8}$$

Again, note that if we denote the sum in the right hand side of (3.8) by w_n $(1 \le n \le N)$, we have

$$\sum_{n=1}^{N} d_n w_n = \sum_{|\alpha| \le N-1} N_\alpha \, (D^\alpha p)(A_P) \,, \tag{3.9}$$

with

$$N_{(0,n-1)} = d_n R(A_P; P) + \sum_{j=n+1}^{N} d_j L_{n-1}^j, 1 \le n \le N \,, \tag{3.10}$$

where $N_{(0,n-1)}$ and L_{n-1}^j are similarly defined. Replacing P by Q in the differentiation formulae (3.5) and (3.8), and using (3.6) and (3.9) in the hypothesis (3.2) we obtain

(3.11)

$$\begin{aligned}&\int_{A_P}^{B_P} R(s,t;P) g' p ds - \sum_{n=0}^{N} c_n \int_{A_Q}^{B_Q} (D_x^n R)(s,t;Q_)g' p \, ds \\ &- \sum_{n=1}^{N} d_n \int_{A_Q}^{B_Q} (D_y^n R)(s,t;Q) g' p \, ds \\ &\qquad\qquad = \sum_{n=1}^{N} c_n v_n - \sum_{n=1}^{N} d_n w_n \\ &\qquad\qquad = \sum_{|\alpha| \le N-1} M_\alpha \, (D^\alpha p)(B_Q) - \sum_{|\alpha| \le N-1} N_\alpha \, (D^\alpha p)(A_Q),\end{aligned}$$

for all polynomials $p(s,t)$. Hence,

(3.12)

$$\int_{A_P}^{B_P} R(s,t;P)g'pds = \int_{A_Q}^{B_Q} R^\star(s,t;Q)g'pds + \sum_{|\alpha|\le N-1} M_\alpha(D^\alpha p)(B_Q) - \sum_{|\alpha|\le N-1} N_\alpha(D^\alpha p)(A_Q),$$

where

$$R^\star(s,t;Q) = \sum_{n=0}^{N} c_n(D_x^n R)(s,t;Q) + \sum_{n=1}^{N} d_n(D_y^n R)(s,t;Q)\,,$$

for all polynomials $p(s,t)$.

Suppose $A_P < A_Q$ (with respect to an obvious order on γ induced by parametrization). Without loss of generality, assume $A_Q \le B_P \le B_Q$. Let T be a point on γ such that $A_P < T < A_Q$. Using that $J_0 > 0$ near the origin and $g' \ne 0$, choose sequence of polynomials $p(s,t)$ such that

$$|\int_{A_P}^{T} Rg'p\,ds| \ge \eta\,,$$

where η is a pre-assigned positive number, while for each α, $0 \le |\alpha| \le N-1$, $|D^\alpha p(s,t)|$ can be made arbitrarily small in $(T, B_Q]$. Using this in (3.12), we get a contradiction. Therefore, we must have $A_Q \le A_P$. Similarly, $B_P \le B_Q$. Hence, (3.11) can be written as

(3.13)

$$\int_{A_Q}^{B_Q} \{R(s,t;P)\chi_{[A_P,B_P]} - \sum_{n=0}^{N} c_n(D_x^n R)(s,t;Q) - \sum_{n=1}^{N} d_n(D_y^n R)(s,t;Q)\}g'pds = \sum_{|\alpha|\le N-1} M_\alpha(D^\alpha p)(B_Q) - \sum_{|\alpha|\le N-1} N_\alpha(D^\alpha p)(A_Q)\,,$$

for all polynomials $p(s,t)$.

Let $A_Q = (a,a')$ and $B_Q = (b,b')$. Consider the sequence of polynomials,

$$p_k(s,t) = \left(\frac{s-a}{b-a}\right)^k, \quad k \ge N\,.$$

Observe that for each k, $(D_s^j p_k)(A_Q) = 0$, $(D_s^j p_k)(B_Q) = 1$ (where D_s^j denotes the j-th derivative with respect to s, $0 \le j \le N-1$), $|p_k(s,t)| < 1$ in $(A_Q,\ B_Q)$ and that $|p_k(s,t)| \to 0$ a.e. on $[A_Q,\ B_Q]$, as $k \to \infty$. Moreover, since for each k, $p_k(s,t)$ does not depend on t (the second variable), all the derivatives of the polynomials p_k with respect to t are zero. Using this sequence of polynomials and applying Lebesgue Dominated Convergence Theorem, we find that the integral in the left hand side of (3.13) goes to zero, as $k \to \infty$. The right hand side of (3.13) reduces to

$$\sum_{n=0}^{N-1} \frac{M_{(n,0)}}{(b-a)^n}\,\frac{k!}{(k-n)!} =: q(k)\,,$$

where $q(x)$ is a polynomial (of degree $\le N-1$). Since the left hand side of (3.13) goes to zero as $k \to \infty$, all coefficients of q must be zero. This immediately implies that $M_{(N-1,0)} = 0$ and a simple argument using induction shows that

$$M_{(n,0)} = 0 \ \forall n \ (0 \le n \le N-1). \tag{3.14}$$

Similarly, replacing $g'(s)ds$ by dt and choosing another sequence of polynomials

$$(\ p_k(s,t) = \left(\frac{t-b'}{a'-b'}\right)^k \ , \ \ k \ge N\ )$$

a similar argument gives

$$N_{(0,n)} = 0 \ \forall n \ (0 \le n \le N-1). \tag{3.15}$$

Using (3.7), (3.14) (respectively, (3.10), (3.15)), the assumption $g' \neq 0$ and the induction we get that $c_i = 0 \ \forall i, \ 1 \le i \le N$ and $d_i = 0 \ \forall i\,, \ 1 \le i \le N$. Hence, $u(P) = c_0 u(Q)$. The required result now follows from Theorem 1.1. □

According to the theorem by L. Schwartz [8, page 165], if T is a distribution supported at the point Q, then we can find unique constants N, c_α ($|\alpha| \le N$) such that

$$T = \sum_{|\alpha| \le N} c_\alpha D^\alpha \delta_Q \ ,$$

where δ_Q denotes the point evaluation at Q.

Using this and Theorem 3.1, we obtain the following partial negative answer to the point to compact set reflection conjecture.

Corollary 3.2. *Let $P \neq Q$ be points in $\mathbb{R}^2$ that are sufficiently close to γ, where γ is a non-singular real analytic curve which is not a straight line. Then, there is no distribution T supported at Q such that $u(P) = \langle T, u\rangle \ \forall u, u_{|_\gamma} = 0$ and satisfying* (1.2).

By using Study's change of variables $z = X + iY$, $w = X - iY$ that reduces the "real" Helmholtz's operator $\triangle + \lambda^2$ to the complex hyperbolic operator $4\frac{\partial^2}{\partial z \partial w} + \lambda^2$, and applying similar arguments we obtain

Corollary 3.3. *Let $P \neq Q$ be points in $\mathbb{R}^2$that are sufficiently close to γ (but off γ), γ being a non-singular real analytic curve in $\mathbb{R}^2$.*

(i) If

$$u(P) = \sum_{|\alpha| \le N} c_\alpha \, D^\alpha u(Q)$$

for all u vanishing on γ and satisfying the real Helmholtz's equation

$$\triangle u + \lambda^2 u = 0\,, \tag{3.16}$$

where $c_\alpha = c_\alpha(P,Q)$ and $N = N(P,Q)$ is an integer, then $c_\alpha = 0 \ \forall \alpha \neq 0$, $c_0 = -1$, P and Q must be symmetric with respect to γ and γ must be a straight line.

(ii) If γ is not a straight line, then there is no distribution T supported at Q such that $u(P) = \langle T, u\rangle \ \forall u, u_{|_\gamma} = 0$ and satisfying (3.16).

Remark 3.4. *For the operator $L = \frac{\partial^2}{\partial s \partial t}$, the Riemann function is identically equal to 1. Hence, equations* (3.4)-(3.10) *are still valid for this case, with $R \equiv 1$ throughout. Hence, arguing as in the proof of Theorem 3.1, we obtain that* (i) *If*

$$u(P) = \sum_{|\alpha| \le N} c_\alpha \, D^\alpha u(Q)$$

for all solutions of $Lu = 0$ *vanishing on* γ*, where* P *and* Q *are two points in* $\mathbb{R}^2$ *that are sufficiently close to* γ*,* $c_\alpha = c_\alpha(P, Q)$ *and* $N = N(P, Q)$ *is an integer, then* $c_\alpha = 0\ \forall\ \alpha \neq 0$*,* $c_0 = -1$ *and* P *and* Q *must be symmetric with respect to* γ*.*

(ii) If

$$\sum_{|\alpha|\leq N} c_\alpha\, D^\alpha u(Q) = 0 \tag{3.17}$$

for all solutions of $Lu = 0$ *vanishing on* γ*, where no mixed derivatives are involved in* (3.17)*,* $c_\alpha = c_\alpha(Q)$ *and* $N = N(Q)$ *is an integer, then* $c_\alpha = 0\ \forall\alpha,\ |\alpha| \leq N$*.*

Similar conclusions hold for $L = \triangle$*.*

As it was mentioned in the introduction of this paper, similar to the case of the two dimensional (real) Helmholtz operator (and when γ is not a straight line), there is no point to point reflection for harmonic functions in $\mathbb{R}^3$ with the Schwarz reflection law, unless the hypersurface under consideration is a hyperplane or a sphere. When the hypersurface is a cylinder, applying our results and using the ideas in the proof of Corollary 3.3 in [6], we obtain the following, which in particular includes that corollary as a special case:

Corollary 3.5. *Let* $P \neq Q$ *be points in* $\mathbb{R}^3$ *that are sufficiently close to* Γ *(but off* Γ*), where*

$$\Gamma = \{(x_1, x_2, x_3) | (x_1, x_2, 0) \in \gamma, \gamma \textit{ being a non-singular real analytic curve}\}$$

is a cylinder in $\mathbb{R}^3$ *with base* γ*.*

(i) If

$$u(P) = \sum_{|\alpha|\leq N} c_\alpha D^\alpha u(Q) \tag{3.18}$$

for all functions u *harmonic near* Γ *and vanishing on* Γ*, where* $c_\alpha = c_\alpha(P, Q)$ *and* $N = N(P, Q)$ *is an integer, then* P *and* Q *must be symmetric with respect to* Γ *and* Γ *must be a plane.*

(ii) If Γ *is not a plane, then there is no distribution* T *supported at* Q *such that*

$$u(P) = \langle T, u\rangle$$

for all u *harmonic near* Γ *and vanishing on* Γ*.*

Proof. (i) Let $P = (x_1^P, x_2^P, x_3^P)$, $Q = (x_1^Q, x_2^Q, x_3^Q)$, $P^0 = (x_1^P, x_2^P)$, $Q^0 = (x_1^Q, x_2^Q)$.

Let $u(x_1, x_2, x_3) = v(x_1, x_2)$, where v is harmonic near γ and vanishing on γ. Then, u is harmonic near Γ and vanishes on Γ. Using (3.18), we obtain,

$$v(P^0) = \sum_{|\alpha|\leq N} c_\alpha D^\alpha v(Q^0),$$

for all v harmonic near γ and vanishing on γ. By Remark 3.4, we obtain that $c_\alpha = 0\ \forall\alpha \neq 0$, $c_0 = -1$ and P^0 and Q^0 must be symmetric with respect to γ. Hence, (3.18) reduces to

$$u(P) = -u(Q) + \sum_{|\alpha|\leq N} c_\alpha (D^\alpha u_{x_3})(Q)\,, \tag{3.19}$$

for all u harmonic near Γ and vanishing on Γ. (Note that from now on, N and the indices of the constants c_α are in general different from those in (3.18)). Next, let

$u(x_1,x_2,x_3)=(x_3-x_3^Q)v(x_1,x_2)$, where v is harmonic near γ and vanishing on γ. Then, u is harmonic near Γ and vanishes on Γ. Using (3.19), we obtain that

$$(x_3^P-x_3^Q)v(P^0)=\sum_{|\alpha|\le N}c_\alpha D^\alpha v(Q_0),$$

for all v harmonic near γ and vanishing on γ. By Remark 3.4, we find that

$$c_\alpha=0\ \forall\alpha\neq 0,\ c_0=-(x_3^P-x_3^Q).$$

Thus, we can write (3.19) (hence, (3.18)) as

$$u(P)=-u(Q)-(x_3^P-x_3^Q)u_{x_3}(Q), \tag{3.20}$$

for all u harmonic near Γ and vanishing on Γ.

Finally, let

$$u(x_1,x_2,x_3)=v(x_1,x_2)e^{\lambda x_3},$$

where,

$$\triangle_{(x_1,x_2)}v+\lambda^2 v=0,\ v_{|_\gamma}=0 \text{ and } \lambda>0\,. \tag{3.21}$$

Then, u is harmonic near Γ and vanishes on Γ. Using (3.20), we find that

$$v(P^0)=-e^{-\lambda(x_3^P-x_3^Q)}(1+\lambda(x_3^P-x_3^Q))v(Q^0)\,,$$

for all v satisfying (3.21). Applying Corollary 3.3 (or Theorem 1.1), we obtain that

$$(1+\lambda(x_3^P-x_3^Q))e^{-\lambda(x_3^P-x_3^Q)}=1\ \forall\lambda\ >0\,, \tag{3.22}$$

and that γ must be a straight line. Hence, Γ must be a plane. Moreover, from (3.22), we must have $x_3^P=x_3^Q$. Hence, since we can assume $x_3^P=x_3^Q=0$, we conclude that P and Q must be symmetric with respect to Γ.

(ii) Follows from (i) and the theorem of L. Schwartz (see the remark preceding Corollary 3.2). □

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